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On a class of N-dimensional associated random walks and the hitting point identity

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N-dimensional random walks on the lattice S with nonnegative integer valued coordinates in $\mathbb{R}^N$ play an important rate in the modelling of computer preformance. For $N>2$ the analysis of those random walks presents difficult analytical problems. The available techniques for $N=2$ are hard to generalize for $N>2$. An interesting identity will be derived for the distribution of the hitting point of the bounding hyperplanes of S. With the aid of this identity it is possible to construct a class of associated N-dimensional random walks on S which are quite accessible for detailed analysis.

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CONTENTS

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INTRODUCTION

In the present study we shall consider random walks in discrete time on the set S of lattice points with integer valued nonnegative coordinates in N-dimensional space $\mathbf{R}^N$. B will indicate the subset of S consisting of all lattice points belonging to the boundary hyperplanes of S. In $S \setminus B$ the random walk will have homogeneous transition probabilities.

Two-dimensional random walks on S have been investigated in the monograph [1], and in that study it has been shown that their analysis may be reduced to that of a functional equation which can be transformed into a standard boundary value problem. Actually this is also possible for $N>2$ but for the resulting boundary value problem no general theory exists and research of those boundary value problems has not yet led to results which are accessible for numerical evaluation.

Recent research of these problems has led to the study of a special class of random walks, to be called 'associated' random walks, for which very explicit results can be obtained. In the derivation of these results the so called 'hitting-point' identity is instrumental, this identity will be derived in Section 4. In the remainder of the present section we shall briefly outline the basic idea of the analysis to be presented.

The structure of the random walk is characterized by its behaviour at the boundary B, i.e. B may be a set of absorbing or reflecting states, and by the stochastic structure of its jump vector which describes the one-step displacements.

In the sequel we shall quite often use *multi-index* notation, it is indicated by using the symbol

"$\equiv$"

in the definitions of the various symbols expressed in this notation.

By ξ will be denoted an N-dimensional stochastic vector with state space S,

$$\xi = \{\xi_1, \ldots, \xi_N\} \in S \equiv \{0,1,2,\ldots\}^N. \tag{1.1}$$

The vector

$$\xi - 1 = \{\xi_1 - 1, \ldots, \xi_N - 1\},$$

with

$$1 \equiv \{1, \ldots, 1\}, \tag{1.2}$$

shall represent the jump vector of the one step displacement of the random walk at points of $S \setminus B$.

With

$$|p_i| \leq 1, \quad i = 1, \ldots, N, \tag{1.3}$$

$$p^\xi \equiv \prod_{i=1}^N p_i^{\xi_i}, \quad p^1 \equiv \prod_{i=1}^N p_i, \tag{1.4}$$

we introduce the N-variate generating function

$$\Phi(p) := E\{p^\xi\} \equiv E\{\prod_{i=1}^N p_i^{\xi_i}\}, \tag{1.5}$$

of the joint distribution of ξ.

Using the notation

$$|p| \leq 1 \equiv \{|p| \leq 1, \ i = 1, \ldots, N\}, \tag{1.6}$$

we introduce the so-called *kernel* $K(p)$,

$$K(p) := p^1 - \Phi(p) \quad \text{for} \quad |p| \leq 1, \tag{1.7}$$

generated by $\Phi(\cdot)$.

A zero tuple $\hat{p}$ of $K(\cdot)$,

$$\hat{p} \equiv (\hat{p}_1, \ldots, \hat{p}_N),$$

is any $\hat{p}$ for which

$$K(\hat{p}) = 0, \quad |\hat{p}| \leqslant 1. \tag{1.8}$$

By x_n we shall denote the position of the random walk at time n, $n=0,1,2,\ldots$; the N-variate generating function of the distribution of x_n is indicated by

$$\Phi_{x_0}^{(n)}(p) := E\{p^{\mathrm{x}_n} | \mathrm{x}_0 = x_0\}, \quad |p| \leqslant 1, \quad n=0,1,\ldots,$$

with $x_0 \in S$ the starting point; we put

$$\Phi_{x_0}(r,p) := \sum_{n=0}^{\infty} r^n E\{p^{\mathrm{x}_n} | \mathrm{x}_0 = x_0\}, \quad |p| \leqslant 1, \quad |r| < 1, \tag{1.9}$$

for the generating function of the sequence $\Phi_{x_0}^{(n)}(p)$, $n=0,1,\ldots$.

The analysis of the random walk requires the solution of a functional equation for $\Phi_{x_0}(r,p)$ of the following type;

For $|r|<1$ and every zero tuple $\hat{p}$ of $p - r\Phi(p)$ determine $\Phi_{x_0}(r,p)$ such that:

i. $\quad L[\Phi_{x_0}(r,\hat{p})] = 0, \qquad (1.10)$

ii. $\quad \Phi_{x_0}(r,p)$ is in each of its variables p_i, regular for $|p_i|<1$, continuous for $|p_i| \leqslant 1$;

here L is a linear functional.

Whenever the boundary B is a reflecting boundary, so that the stochastic structure of the random walk x_n, $n=0,1,\ldots$, is described by: for $n=0,1,2,\ldots$,

$$\mathrm{x}_{n+1} = [\mathrm{x}_n - 1]^+ + \xi^{(n)}, \tag{1.11}$$

$$\mathrm{x}_0 = x_0,$$

$\xi^{(n)}$, $n=0,1,2,\ldots$, being i.i.d. vectors, $\xi^{(n)} \sim \xi$; then the random walk may possess a stationary distribution. Suppose it does, and denote by x a stochastic vector with distribution this stationary distribution then

$$\Phi(p) = E\{p^{\mathrm{x}}\}, \quad |p| \leqslant 1, \tag{1.12}$$

should satisfy (1.10) with $\hat{p}$ a zero tuple of $K(\cdot)$, cf. (1.8). The solution of (1.10) for this case depends essentially on the analytical representation of the zero tuples of $K(\cdot)$. Such a representation is fairly simple if the vector ξ has independent components, i.e. if

$$\Phi(p) = \prod_{i=1}^{N} \Phi_i(p_i), \quad |p_i| \leqslant 1, \quad \Phi_i(p_i) := E\{p_i^{\xi}\}. \tag{1.13}$$

If ξ has independent components then it is not difficult to construct from (1.10) the explicit expression for $\Phi(p)$, cf. (1.12). This construction evidently depends essentially on the zero tuples of $K(\cdot)$. It is, therefore, natural to investigate the question: does there exist a vector η with N-variate generating function $\omega(\cdot)$, such that the zero tuples of $K(\cdot)$, cf. (1.7), are also zero tuples of

$$p^1 - \omega(p), \quad |p| \leqslant 1. \tag{1.14}$$

If so the functional equations (1.10) pertaining to the random walk (1.11) with ξ replaced by η can be solved, i.e. the analytic description of this random walk can be obtained explicitly.

Indeed such vectors η, to be called *associated with* ξ, do exist. An example of such a vector is the vector x which is the hitting point of B when B is an absorbing boundary for the random walk x_n with displacement vector ξ and starting point $\mathrm{x}_0 = 1$. Actually the class of vectors associated with ξ is quite large and as such may lead to interesting possibilities for the analysis of N-dimensional random walks, in particular for those occurring in the analysis of queueing systems which are encountered in performance analysis of computer networks.

Next we shall briefly review the various sections. In Section 2 we formulate the random walk $\mathbf{x}_n$ on S with the boundary B of S a set of absorbing states, here we also introduce the multi-index notation and the notation for the entrance times and hitting points of the bounding hyperplanes of S; the boundary B is the union of these hyperplanes. In Section 3 the kernel of the random walk is introduced, this kernel is directly related to the structure of the random walk on $S \setminus B$. In Section 4 the *hitting point identity* is formulated. This identity is a relation between the zero tuples of the kernel and the generating function of the distribution of the hitting point of B. This identity is a most interesting one, see also [3], [4] and [5] for some applications. In the present study this identity is used to construct associated jump vectors, see Sections 6,7 and 9. The resolution of the hitting point identity is discussed in Section 5 for the case that the jump vector ξ of the random walk $\mathbf{x}_n$ has independent components. In Section 8 a reflecting random walk on S is introduced. For this random walk the functional equation for its generating function is derived, and discussed in detail for the case that the jump vector ξ has independent components. The results obtained in this section are instrumental for the analysis of associated random walks. That analysis is discussed in Section 9. It turns out that a most interesting relation exists between the generating functions of the stationary distributions of a random walk and an associated one of it. In Section 10 attention is paid to the explicit construction of associated jump vectors; three examples are discussed, and for one of them the associated random walk is investigated in some detail in Section 11.

2. The absorbing random walk

The random walk $\mathbf{x}_n$, $n=0,1,2,\ldots$, is defined by

$$\begin{aligned} \mathbf{x}_0 &= x_0 \in S, \\ \mathbf{x}_{n+1} &= \mathbf{x}_n - 1 + \xi^{(n)}, \quad \text{for } \mathbf{x}_n \in S \setminus B, \\ &= \mathbf{x}_n, \qquad\qquad\quad \text{for } \mathbf{x}_n \in B, \end{aligned} \tag{2.1}$$

with $\xi^{(n)}$, $n=0,1,2,\ldots$, i.i.d. vectors and

$$\xi^{(n)} \sim \xi, \quad \phi(p) = E\{p^{\xi_\cdot}\} = E\{p^{\xi}\}, \quad |p| \leq 1. \tag{2.2}$$

So x_0 is the starting point and the boundary B is absorbing.

We first introduce some notation to specify the bounding hyperplanes of S.

Let $\mathfrak{N}$ be the set of vectors $b=(b_1,\ldots,b_N)$,

$$\mathfrak{N} := \{b \in S;\ b_m \in (0,1),\ m=1,\ldots,N;\ \prod_{j=1}^{N} b_j = 0\}, \tag{2.3}$$

so the cardinal number of $\mathfrak{N}$ is given by

$$|\mathfrak{N}| = 2^N - 1. \tag{2.4}$$

Put for $b \in \mathfrak{N}$,

$$B(b) := \{j \in S:\ j_m = 0 \text{ if } b_m = 0;\ j_m > 0 \text{ if } b_m = 1;\ m=1,\ldots,N\}, \tag{2.5}$$

so that for $b,c \in \mathfrak{N}$, and $b \neq c$,

$$\begin{aligned} B(b) \cap B(c) &= \varnothing, \\ B &= \bigcup_{b \in \mathfrak{N}} B(b), \end{aligned} \tag{2.6}$$

and

$$1 - b \equiv (1-b_1,\ldots,1-b_N), \quad b \in \mathfrak{N}, \tag{2.7}$$

$$ib \equiv (ib_1, \ldots, i_N b_N) \in B(b), \quad i \in S, \quad b \in \mathfrak{N}.$$

Further we introduce for the random walk $\mathbf{x}_n$, $n=0,1,2,\ldots,$ with $x_0 \in S \setminus B$:

i. $\mathbf{n}(x_0)$ the hitting time of B, (2.8)

ii. $\mathbf{n}_b(x_0)$ the hitting time of $B(b)$,

iii. $\mathbf{k}(x_0) \equiv (\mathbf{k}^{(1)}(x_0), \ldots, \mathbf{k}^{(N)}(x_0))$ the hitting point of B,

iv. $\mathbf{k}_b(x_0) \equiv (\mathbf{k}_b^{(1)}(x_0), \ldots, \mathbf{k}_b^{(N)}(x_0))$ the hitting point of $B(b)$, $b \in \mathfrak{N}$, if $\mathbf{n}_b(x_0) < \infty$, so $\mathbf{k}_b^{(i)}(x_0) = 0$ if $b_i = 0$,

iv. $a_b(x_0) = Pr\{\mathbf{n}_b(x_0) < \infty\}$,

$a(x_0) = Pr\{\mathbf{n}(x_0) < \infty\}$.

For $x_0 \in B$ we make the convention:

i. $\mathbf{n}(x_0) := 0,$ (2.9)

$\mathbf{n}_b(x_0) := 0$ if $x_0 \in B_b, \quad b \in \mathfrak{N},$

$:= \infty$ if $x_0 \notin B_b$,

ii. $\mathbf{k}(x_0) = x_0,$

$\mathbf{k}_b(x_0) = x_0$ if $x_0 \in B_b$,

$a_b(x_0) = 1$ if $x_0 \in B(b)$,

$= 0$ if $x_0 \in B \setminus B_b$;

iii. $(\mathbf{x}_n \in A) := 1\{\mathbf{x}_n \in A\}$ with $A \in S$,

shall indicate the indicator function of the event $\{\mathbf{x}_n \in A\}$;
if $x_0 = 1$ then we define for later use

$$\mathbf{n} := 1+\mathbf{n}(1), \quad \mathbf{n}_b := 1+\mathbf{n}_b(1), \quad a_b \equiv a_b(1), \quad \mathbf{k}_b = \mathbf{k}_b(1). \tag{2.10}$$

For $x_0 \in S$, $|p| \leq 1$, put

$$\Phi_{x_0}^{(n)}(p) := E_{x_0}\{p^{\mathbf{x}_n}\} = E\{p^{\mathbf{x}_n} | \mathbf{x}_0 = x_0\}, \tag{2.11}$$

$$\Phi_{x_0}^{(n)}(p,b) := E_{x_0}\{p^{\mathbf{x}_n}(\mathbf{x}_n \in B(b))\}, \quad b \in \mathfrak{N}.$$

It follows from (2.1) and (2.2) that $\mathbf{x}_n$ and $\boldsymbol{\xi}_n$ are independent, and this observation leads easily to: for $|p| \leq 1$, $\mathbf{x}_0 \in S$, $n=0,1,\ldots,$

$$\Phi_{x_0}^{(n+1)}(p) = E\{p^{\boldsymbol{\xi}^{(n)}}\}\Phi_{x_0}^{(n)}(p) + [1 - E\{p^{\boldsymbol{\xi}^{(n)}}\}] \sum_{b \in \mathfrak{N}} \Phi_{x_0}^{(n)}(p,b). \tag{2.12}$$

Since

$$|\Phi_{x_0}^{(n)}(p)| \leq 1, \quad |\Phi_{x_0}^{(n)}(p,b)| \leq 1, \quad |p| \leq 1,$$

we may define for $|r| < 1$, $|p| \leq 1$, $b \in \mathfrak{N}$, $x_0 \in S$,

$$\Phi_{x_0}(r,p) := \sum_{n=0}^{\infty} r^n \Phi_{x_0}^{(n)}(p), \tag{2.13}$$

$$\Phi_{x_0}(r,p,b) := \sum_{n=0}^{\infty} r^n \Phi_{x_0}^{(n)}(p,b).$$

From (2.12) and (2.13) we obtain: for $|p| \leq 1, |r| < 1$, $x_0 \in S$,

$$[1-rE\{p^{\xi-1}\}]\Phi_{x_0}(r,p) = p^{x_0}+r[1-E\{p^{\xi-1}\}]\sum_{b\in\mathfrak{N}}\Phi_{x_0}(r,p,b). \tag{2.14}$$

We now have

LEMMA 2.1. *For* $|p|\leq 1, |r|<1$, $x_0\in S$,

$$[p^1-rE\{p^{\xi}\}]\Phi_{x_0}(r,p) = p^{x_0+1}+\frac{r}{1-r}[p^1-E\{p^{\xi}\}]\sum_{b\in\mathfrak{N}}E\{r^{\mathbf{n}_b(x_0)}p^{\mathbf{k}_b(x_0)}\}. \tag{2.15}$$

PROOF. Because B is an absorbing boundary it follows by using the definition above that

$$\begin{aligned}\{\mathbf{x}_n\in B(b)\} &= \{\mathbf{x}_{n+k}\in B(b)\} \quad \text{for} \quad k=0,1,2,\ldots,\\ &= \{\mathbf{n}_b(x_0)\leq n,\ \mathbf{k}_b(x_0)=\mathbf{x}_n\in B_b\}.\end{aligned}$$

Hence

$$\Phi_{x_0}^{(n)}(p,b) = E_{x_0}\{p^{\mathbf{x}_n}(\mathbf{x}_n\in B_b)\} = E\{p^{\mathbf{k}_b(x_0)}(\mathbf{n}_b(x_0)\leq n)\},$$

and

$$\Phi_{x_0}(r,p,b) = \sum_{n=0}^{\infty}r^nE\{p^{\mathbf{k}_b(x_0)}(\mathbf{n}_b(\mathbf{x}_0)\leq \underline{n})\} = \frac{1}{1-r}E\{r^{\mathbf{n}_b(x_0)}p^{\mathbf{k}_b(x_0)}\},$$

and the lemma follows from (2.14). □

Next we formulate

LEMMA 2.2. *For* $|r|<1$, $|p|\leq 1$, $x_0\in S$.

$$[p^1-rE\{p^{\xi}\}]\sum_{n=0}^{\infty}r^nE_{x_0}\{p^{\mathbf{x}_n}(\mathbf{x}_n\notin B)\} = p^1[p^{x_0}-E\{r^{\mathbf{n}(x_0)}p^{\mathbf{k}(x_0)}\}]. \tag{2.16}$$

PROOF. We have from (2.11) and (2.13) for $|r|<1$, $|p|\leq 1$,

$$\Phi_{x_0}(r,p)-\sum_{n=0}^{\infty}r^nE\{p^{\mathbf{x}_n}(\mathbf{x}_n\notin B)\} = \sum_{n=0}^{\infty}r^nE\{p^{\mathbf{x}_n}(\mathbf{x}_n\in B)\}= \tag{2.17}$$

$$\sum_{b\in\mathfrak{N}}\Phi_{x_0}(r,p,b) = \frac{1}{1-r}\sum_{b\in\mathfrak{N}}E\{r^{\mathbf{n}_b(x_0)}p^{\mathbf{k}_b(x_0)}\} = \frac{1}{1-r}E\{r^{\mathbf{n}(x_0)}p^{\mathbf{k}(x_0)}\}.$$

Hence from (2.15)

$$[p^1-rE\{p^{\xi}\}]\Phi_{x_0}(r,p) = [p^1-rE\{p^{\xi}\}]\{\sum_{n=0}^{\infty}r^nE_{x_0}\{p^{\mathbf{x}_n}(\mathbf{x}_n\notin B)\}+$$

$$\frac{1}{1-r}E\{r^{\mathbf{n}(x_0)}p^{\mathbf{k}(x_0)}\}=p^{x_0+1}+\frac{r}{1-r}[p^1-E\{p^{\xi}\}]E\{r^{\mathbf{n}(x_0)}p^{\mathbf{k}(x_0)}\}. \tag{2.18}$$

Rearranging the terms in the last equality of (2.18) leads directly to (2.16).

3. THE KERNEL

The kernel $K(r,p)$ generated by ξ, cf. (1.1) and (1.5), is defined by:

$$\begin{aligned}K(r,p) &:= p^1-rE\{p^{\xi}\}=p^1-r\phi(p), \quad |p|\leq 1 \quad \text{for} \quad |r|<1,\\ K(p) &:= K(1,p), \quad |p|\leq 1 \quad \text{for} \quad r=1,\end{aligned} \tag{3.1}$$

and

$$\hat{p} = (\hat{p}_1, \ldots, \hat{p}_N), \quad |\hat{p}| \leqslant 1, \tag{3.2}$$

is called a zero tuple of $K(r,p)$, $|r| \leqslant 1$, if

$$K(r,\hat{p}) = 0.$$

LEMMA 3.1. *For every fixed* $|r| \leqslant 1$ *the kernel* $K(r,p)$ *possesses a nonempty set of zero tuples.*

PROOF. Obviously $\hat{p} = 0$ is a zero tuple if $r = 0$, so suppose $r \neq 0$.
Take

$$p_m = g s_m, \quad |s_m| = 1, \quad m = 1, \ldots, N, \tag{3.3}$$

with

$$\prod_{m=1}^{N} s_m = 1, \quad |g| \leqslant 1. \tag{3.4}$$

Then

$$K(r,p) = g^N - rE\{g^{\Sigma_{i=1}^N \xi_i} \prod_{i=1}^{N} s_i^{\xi_i}\}. \tag{3.5}$$

From (3.3) and (3.4) it follows for $|r| < 1$ that for $|g| = 1$,

$$|g^N| = 1 > |r| E\{|g|^{\Sigma_{i=1}^N \xi_i}\} \geqslant |rE\{g^{\Sigma_{i=1}^N \xi_i} \prod_{i=1}^{N} s_i^{\xi_i}\}|. \tag{3.6}$$

The second term in (3.5) is regular for $|g| < 1$, continuous for $|g| \leqslant 1$ and so is the last term of (3.5) for fixed r and s_i, $i = 1, \ldots, N$. Hence by applying Rouché's theorem it follows that the right-hand side of (3.5) has exactly N zeros counted according to their multiplicity in $|g| \leqslant 1$. With every such zero corresponds according to (3.3) a zero tuple of $K(r,p)$, and the lemma is proved for $|r| < 1$. For $|r| = 1$ the proof follows easily from the lemma for $|r| < 1$ by using a continuity argument and letting $|r| \to 1$, see also [1]. □

For fixed $i \in \{1, \ldots, N\}$ denote by

$$\mu_i(r), \quad |r| < 1, \tag{3.7}$$

the unique zero of

$$p_i - rE\{p_i^{\xi_i}\}, \quad |p_i| \leqslant 1. \tag{3.8}$$

By applying Rouché's theorem the existence and uniqueness is easily proved and it is well known that a stochastic variable $\mathbf{m}_i$ exists with range space $\{1,2,\ldots\}$ such that

$$\mu_i(r) = E\{r^{\mathbf{m}_i}\}, \quad |r| \leqslant 1, \tag{3.9}$$

and

$$\begin{aligned} \mu_i(1-) = Pr\{\mathbf{m}_i < \infty\} &= 1 \text{ if } E\{\xi_i\} \leqslant 1, \\ &< 1 \text{ if } \qquad > 1, \\ E\{\mathbf{m}_i < \infty\} &< \infty \text{ if } E\{\xi_i\} < 1, \\ &= \infty \text{ if } \qquad = 1. \end{aligned} \tag{3.10}$$

It is also well known that $\mathbf{m}_i$ has an aperiodic distribution if and only if ξ_i has such a distribution, i.e.

$$g.c.d.\{k: Pr\{\xi_i = k\} > 0\} = 1,$$

or equivalently, cf. [2],

$$|p_i| = 1 \quad \text{and} \quad |E\{p_i^{\xi}\})=1 \Leftrightarrow p_i=1.$$

From now on, i.e. in this and all the following sections we make concerning ξ the

ASSUMPTION 3.1.

i. $$E\{\xi_i\}<1, \quad i=1,\dots,N; \phi(0)=E\{\prod_{i=1}^{N}(\xi_i=0)\}>0; \tag{3.11}$$

ii. $|\phi(p)|=1$ & $|p_i|=1$, $i=1,\dots,N \Rightarrow p_i=1$, $i=1,\dots,N$;

iii. for every $i \in S$ the coefficient of p^i in the series development of $[\phi(p)/p]^n$ is positive for n sufficiently large.

REMARK 3.1. The assumption (3.11)ii it is introduced to guarantee that if $\hat{p}$ is a zero tuple of $K(p)$, cf. (3.1), with $|\hat{p}_i|=1$ then $\hat{p}=1$, whereas (3.11)i implies that if $\hat{p}$ is a zero tuple of $K(p)$, with $N-1$ of its components p_i equal to one then the remaining component is necessarily equal to one. Both these consequences are easily proved. For an analysis without the introduction of (3.11)i see [3]. The condition (3.11)iii is introduced to guarantee that the reflecting random walks to be studied have an irreducible state space; for the absorbing random walk (2.1), (3.11)iii is irrelevant.

REMARK 3.2. By taking again the parametrisation (3.3) and (3.4) we obtain to $r=1$:

$$K(p) = g^N - E\{g^{\Sigma_{i=1}^{N}\xi_i}\prod_{i=1}^{N}s_i^{\xi_i}\}. \tag{3.12}$$

By noting that (3.11) implies $E\{\xi_i+\cdots+\xi_n\}<N$ if follows from well-known arguments that the right-hand side of (3.12) has exactly N zeros in $|g|\leq 1$.

4. THE HITTING-POINT IDENTITY

THEOREM 4.1. *For $\hat{p}(r)$, $|r|<1$ a zero tuple of $K(r,p)$ and $x_0 \in S$:*

i. $$\hat{p}^{x_0}(r) = \sum_{b\in\mathfrak{N}} E\{r^{\mathbf{n}_b(x_0)}\hat{p}^{\mathbf{k}_b(x_0)}(r)\}, \quad x_0 \in S, \tag{4.1}$$

ii. $$\hat{p}^{1}(r) = \sum_{b\in\mathfrak{N}} E\{r^{\mathbf{n}_b}\hat{p}^{\mathbf{k}_b}(r)\},$$

iii. $Pr\{\mathbf{n}(x_0)<\infty\}=1$, $Pr\{\mathbf{n}<\infty\}=1$.

PROOF. Because

$$|\Phi_{x_0}(r,p)| \leq \frac{1}{1-r} \quad \text{for} \quad |r|<1, \quad |p|\leq 1,$$

it follows from (2.15) and (3.1) that the right-hand side of (2.15) should be zero for any zero tuple $\hat{p}(r)$ of $K(r,p)$ with $|r|<1$, and this leads immediately to (4.1)i.

To prove (4.1)ii we take for the starting point x_0 of the absorbing random walk $\mathbf{x}_n$, cf. (2.1), a random point $\mathbf{x}_0$ with distribution that of ξ, so that by using the definition of $\hat{p}(r)$ we obtain from (4.1)i and (2.10) for $|r|<1$,

$$\hat{p}^{1}(r) = rE\{\hat{p}^{\mathbf{x}_0}(r)\} = \sum_{j\in S} r \sum_{b\in\mathfrak{N}} E\{r^{\mathbf{n}_b(j)}\hat{p}^{\mathbf{k}_b(j)}(r)\}Pr\{\mathbf{x}_0=j\} = E\{r^{\mathbf{n}}\hat{p}^{\mathbf{k}_b}(r)\},$$

which proves (4.1)ii.

To prove (4.1)iii take for $|r|<1$,

$$\hat{p}_i(r) = 1 \quad \text{for} \quad i=2,\dots,N,$$

then it follows readily from (3.11)i and from (3.8), ..., (3.10) for $|r|<1$ that $\hat{p}_1(r)=E\{r^{\mathbf{m}_1}\}$. Hence from (4.1)ii for $|r|<1$:

$$E\{r^{\mathbf{m}_1}\} = \sum_{b\in\mathfrak{N}} E\{r^{\mathbf{n}_b}\hat{p}_1^{\mathbf{k}_b^{(1)}}(r)\prod_{i=2}^{N}\hat{p}_i^{\mathbf{k}_b^{(i)}}(r)\} = \sum_{b\in\mathfrak{N}} E\{r^{\mathbf{n}_b}[E\{r^{\mathbf{m}_1}\}]^{\mathbf{k}_b^{(1)}}\}.$$

So that, cf. (3.10),

$$1 = \lim_{r\uparrow 1} E\{r^{\mathbf{m}_1}\} = \sum_{b\in\mathfrak{N}} E\{(\mathbf{n}_b<\infty)\} = E\{(\mathbf{n}<\infty)\},$$

and this proves the second statement of (4.1)iii; the first one follows by the same argument from (4.1)i. □

THEOREM 4.2. *The hitting-point identity. For $\hat{p}$ a zero tuple of the kernel $K(p)$,*

$$\hat{p}^{\,1} = \sum_{b\in\mathfrak{N}} a_b E\{\hat{p}^{\mathbf{k}_b}|\mathbf{n}_b<\infty\}; \tag{4.2}$$

for $n=0,1,2,\ldots$; $x_0\in S$,

$$\hat{p}^{\,x_0} = \sum_{b\in\mathfrak{N}} E\{\hat{p}^{\mathbf{k}_b(x_0)}(\mathbf{n}_b(x_0)\leq n)\} + E\{\hat{p}^{\mathbf{x}_n}(\mathbf{n}(x_0)>n)|\mathbf{x}_0=x_0\}, \tag{4.3}$$

$$= \sum_{b\in\mathfrak{N}} a_b(x_0)E\{\hat{p}^{\mathbf{k}_b(x_0)}|\mathbf{n}_b(x_0)<\infty\}.$$

PROOF. From Lemma 3.1 see also Remark 3.2, we know that $K(p)$ possesses zero tuples. Let $\hat{p}$ be a zero tuple, so

$$\hat{p}^{\,1} = E\{\hat{p}^{\xi}\},$$

it then follows from (2.15) with $|r|<1$,

$$\hat{p}^{\,x_0} = (1-r)\Phi_{x_0}(r,\hat{p}) = (1-r)\sum_{n=0}^{\infty} r^n E\{\hat{p}^{\mathbf{x}_n}|\mathbf{x}_0=x_0\},$$

i.e.

$$\sum_{n=0}^{\infty} r^n E\{\hat{p}^{\mathbf{x}_n}|\mathbf{x}_0=x_0\} = \hat{p}^{\,x_0}\sum_{n=0}^{\infty} r^n. \tag{4.4}$$

By equating coefficients of r^n in (4.4) it follows that for every $n=1,2,\ldots$,

$$\hat{p}^{\,x_0} = E\{\hat{p}^{\mathbf{x}_n}|\mathbf{x}_0=x_0\} = \sum_{b\in\mathfrak{N}} E\{\hat{p}^{\mathbf{x}_n}(\mathbf{x}_n\in B(b))|\mathbf{x}_0=x_0\} + E\{\hat{p}^{\mathbf{x}_n}(\mathbf{x}_n\in S\setminus B)|\mathbf{x}_0=x_0\} \tag{4.5}$$

$$= \sum_{b\in\mathfrak{N}} E\{\hat{p}^{\mathbf{k}_b(x_0)}(\mathbf{n}_b(x_0)\leq n)\} + E\{\hat{p}^{\mathbf{x}_n}(\mathbf{n}(x_0)>n)|\mathbf{x}_0=x_0\},$$

and this proves (4.3). Since (4.5) holds for every $n=1,2,\ldots$, and $|\hat{p}|\leq 1$ it follows from (4.1)iii,

$$\hat{p}^{\,x_0} = \sum_{b\in\mathfrak{N}} E\{\hat{p}^{\mathbf{k}_b(x_0)}(\mathbf{n}_b(\mathbf{x}_0)<\infty)\}, \quad x_0\in S. \tag{4.6}$$

By noting that (4.6) holds for every $x_0\in S$, it follows by taking x_0 a stochastic variable with distribution that of ξ, cf. (2.10),

$$\hat{p} = E\{\hat{p}^{\mathbf{x}_0}\} = \sum_{b\in\mathfrak{N}} E\{\hat{p}^{\mathbf{k}_b}(\mathbf{n}_b<\infty)\} = \sum_{b\in\mathfrak{N}} a_b E\{\hat{p}^{\mathbf{k}_b}|\mathbf{n}_b<\infty\}.$$

and this proves (4.2). □

REMARK 4.1. If the initial point x_0 of the random walk $\mathbf{x}_n$ is a stochastic variable with distribution that of ξ then this random walk may be considered to be started at $(0,0,\dots,0)$ with an initial jump ξ. Hence the first jump may then lead to a first entrance into $B\setminus\{0,\dots,0\}$, and from ass. (3.11)ii it is seen that the distribution of every coordinate at $\mathbf{k}(\xi)$, cf. (2.8)iii, is aperiodic.

THEOREM 4.3. *i. For p, with $|p|\leq 1$, not a zero tuple of $K(p)$:*

$$\sum_{n=0}^{\infty} E_{x_0}\{p^{\mathbf{x}_n}(\mathbf{x}_n\notin B)\} = p^1\frac{p^{x_0}-E\{p^{\mathbf{k}(x_0)}\}}{p-E\{p^{\xi}\}},\quad x_0\in S, \tag{4.7}$$

$$ii.\quad \infty>E\{\mathbf{n}(1)\} = \frac{1-E\{\mathbf{k}^{(i)}(1)\}}{1-E\{\xi_i\}}>1,\ i=1,\dots,N;\ \xi=\{\xi_1,\dots,\xi_N\}, \tag{4.8}$$

$$0<E\{\mathbf{k}^{(i)}(1)\}<E\{\xi_i\},\quad i=1,\dots,N. \tag{4.9}$$

PROOF. By noting that $\mathbf{n}(x_0)$ is finite with probability one, cf. theorem 4.1, the relation (4.7) follows directly from lemma 2.2 by letting $r\uparrow 1$.

To prove (4.8) take in (4.7) $x_0=1$ and

$$|p_1|\leq 1, p_1\neq 1,\ p_j=1,\ j=2,\dots,N,$$

then

$$\sum_{n=0}^{\infty} E_1\{p^{\mathbf{x}_n^{(1)}}(\mathbf{x}_n\in B)\} = p_1\frac{p_1-E\{p_1^{\mathbf{k}^{(1)}(1)}\}}{p_1-E\{p_1^{\xi_1}\}}. \tag{4.10}$$

By noting that, since $x_0=1\notin B$,

$$(x_0\notin B) = 1,$$

$$(\mathbf{x}_n\notin B) = (\mathbf{n}(1)>n),\quad n=1,2,\dots,$$

it follows by letting $p_1\uparrow 1$ that

$$\frac{1-E\{\mathbf{k}^{(1)}(1)\}}{1-E\{\xi_1\}} = \sum_{n=0}^{\infty}E\{(\mathbf{x}_n\notin B)\}=1+\sum_{n=1}^{\infty}E\{(\mathbf{n}(1)>n)\}= \tag{4.11}$$

$$1+\sum_{n=1}^{\infty}E\{(\mathbf{n}(1)\geq n)\}-E\{\sum_{n=1}^{\infty}(\mathbf{n}(1)=n)\}=1+E\{\mathbf{n}(1)\}-1.$$

Since (3.11)i implies that $Pr\{\mathbf{n}(1)=1\}<1$ the relation (4.8) follows from (4.11) for $i=1$, similarly for $i=2,\dots,N$, since $\mathbf{n}(1)\geq 1$ with probability one. The relation (4.9) follows directly from (4.8). □

REMARK 4.2. Theorem 4.1 states that every zero tuple of $p^1-rE\{p^{\xi}\}$ is also a zero tuple of $p^1-E\{r^{\mathbf{n}(1)}p^{\mathbf{k}(1)}\}, |r|\leq 1$. The converse statement is also true. For such a statement see corollary 9.1. A direct proof can be given by starting from the relation (2.16) with $x_0=1$.

5. RESOLUTION OF THE IDENTITY FOR THE INDEPENDENT CASE

From Theorem 4.2 we see that

$$a_b E\{p^{\mathbf{k}_b}|\mathbf{n}_b<\infty\},\quad b\in\mathfrak{N},\quad |p|\leq 1,$$

satisfy for every zero tuple $\hat{p}$ of $K(p)$ the hitting point identity. It may be shown by similar arguments as have been used in [3], (cf. there Theorem 6.2) that the quantities in (5.1) are uniquely determined by the conditions: i. they should satisfy (4.3) for every zero tuple of $K(p)$ and ii. they should be regular in each $|p_i|<1$, continuous in each $|p_i|\leq 1$. In general, i.e. for $N\geq 3$, it is difficult to resolve the hitting point identity (4.3), i.e. to construct explicit expressions for the terms occurring in this identity. However, if ξ has independent components such a resolution is possible for every N. Below we discuss this case.

Suppose that $\xi=\{\xi_1,\ldots,\xi_N\}$ has independent components, so that we may and do write

$$\phi_i(p_i) := E\{p_i^{\xi_i}\}, \quad i=1,\ldots,N. \tag{5.1}$$

$$\phi(p) = \prod_{i=1}^{N}\phi_i(p_i) = E\{p^{\xi}\}, \quad |p|\leq 1.$$

By noting assumption (3.11)i and ii. and by using Rouché's theorem it follows, cf. (3.7), . . . , (3.10), that the function

$$p_i - t_i\phi_i(p_i), \quad |p_i|\leq 1, \tag{5.2}$$

with $|t_i|\leq 1$, has exactly one solution

$$p_i = \mu_i(t_i) := E\{t_i^{\mathbf{m}_i}\}, \quad |t_i|\leq 1, \tag{5.3}$$

with

$$\mathbf{m}_i\in\{1,2,\ldots\}, \tag{5.4}$$

$$Pr\{\mathbf{m}_i<\infty\} = 1, \quad E\{\mathbf{m}_i\} = \frac{1}{1-E\{\xi_i\}}.$$

Consequently: with

$$|t_i|=1, \quad i=1,..,N, \quad \prod_{i=1}^{N}t_i=1, \tag{5.5}$$

it is seen that

$$\hat{p} := (\mu_1(t_1),\ldots,\mu_N(t_N)), \tag{5.6}$$

is a zero tuple of $K(p)$.

By using the parametrization (5.6) for a class of zero tuples of the kernel $K(p)$ it is now possible to resolve the hitting point identity. This resolution is based on complex integration, and for this we need the

LEMMA 5.1. *For the $\mu_i(\cdot)$ as defined above:*

i. $\mu_i(\tau_i)$ *is regular for* $|\tau_i|<1$, *continuous for* $|\tau_i|\leq 1$, $i=1,\ldots,N-1$; (5.7)

ii. $\mu_N(\dfrac{1}{\tau_1\cdots\tau_{N-1}})$ *is for fixed* τ_k *with* $|\tau_k|\geq 1$, $k=2,\ldots,N-1$;

regular in τ_1 *for* $|\tau_1|>1$, *continuous for* $\tau_1\geq 1$;

iii. $\mu_i(\tau_i) = 0$ *for* $\tau_i = 0$, $i=1,\ldots,N$.

PROOF. The proof follows directly from (5.3). □

We now apply the integral operator

$$\{\prod_{j=1}^{N-1}\frac{1}{2\pi i}\int_{|\tau_j|=1}\frac{d\tau_j}{\tau_j-t_j}\}\ldots \quad \text{with } |t_j|<1, \quad j=1,\ldots,N-1, \tag{5.8}$$

to the hitting point identity, with $\hat{p}$ as given by (5.5) and (5.6), i.e. for $|t_j|<1$, $j=1,\ldots,N-1$,

$$\{\prod_{j=1}^{N-1}\frac{1}{2\pi i}\int_{|\tau_j|=1}\frac{\mu_j(\tau_j)}{\tau_j-t_j}\}\mu_N(\frac{1}{\tau_1\cdots\tau_{N-1}})d\tau_1\cdots d\tau_{N-1}= \tag{5.9}$$

$$= \sum_{b\in\mathfrak{N}} a_b\{\prod_{j=1}^{N-1}\frac{1}{2\pi i}\int_{|\tau_j|=1}\frac{d\tau_j}{\tau_j-t_j}\}E\{[\mu_1(\tau_1)]^{b_1\mathbf{k}_b^{(1)}}\cdots[\mu_{N-1}(\tau_{N-1})]^{b_{N-1}\mathbf{k}_b^{(N-1)}}\times$$
$$\times[\mu_N(\frac{1}{\tau_1\cdots\tau_{N-1}})]^{b_N\mathbf{k}_b^{(N)}}|\mathbf{n}_b<\infty\}.$$

We remember that , cf. (2.3),

$$\prod_{i=1}^{N}b_i = 0, \quad \text{for} \quad b\in\mathfrak{N}. \tag{5.10}$$

Suppose $b_1=0, b_N=1$ so that the integration variable τ_1, occurs in the integrand only in the factor

$$\frac{1}{\tau_1-t_1}[\mu_n(\frac{1}{\tau_1\cdots\tau_{N-1}})]^{b_N\mathbf{k}_b^{(N)}}, \quad |\tau_1|=1, \tag{5.11}$$

then from Lemma 5.1 ii and iii it follows by contour integration outside $|\tau_1|=1$, where the function in (5.11) is regular, note $|\tau_1|<1$, and tends sufficiently rapidly to zero for $|\tau_1|\to\infty$, that

$$\frac{1}{2\pi i}\int_{|\tau_1|=1}\frac{d\tau_1}{\tau_1-t_1}[\mu_N(\frac{1}{\tau_1\cdots\tau_{N-1}})]^{b_N\mathbf{k}_b^{(N)}}=0, \quad |\tau_j|=1, \quad j=2,\ldots,N-1, \tag{5.12}$$

note $\mathbf{k}_b^{(N)}>0$.

Consequently it follows that the coefficient of a_b in the right-hand side of (5.9) is zero if $b_N=1$.

If $b_N=0$ then the coefficient of a_b is the product of $N-1$ integrals, each integrand being regular inside the unit circle except for a single pole, and by Cauchy's theorem such an integral is equal to its residue at that pole.

Hence it follows with

$$\mathfrak{N}_{b_N=0}:=\{b: b\in\mathfrak{N}, b_N=0\}, \tag{5.13}$$

that for $|t_j|<1, j=1,\ldots,N-1$,

$$\sum_{b\in\mathfrak{N}_{b_N=0}} a_bE\{\prod_{j=1}^{N-1}[\mu_j(t_j)]^{b_j\mathbf{k}_b^{(j)}}|(\mathbf{n}_b<\infty)\} = \{\prod_{j=1}^{N-1}\frac{1}{2\pi i}\int_{|\tau_j|=1}\frac{\mu_j(\tau_j)}{\tau_j-t_j}\}\mu_N(\frac{1}{\tau_1\cdots\tau_{N-1}})d\tau_1\cdots d\tau_{N-1}. \tag{5.14}$$

Put for $|t_j|<1, j=1,\ldots,N-1$,

$$I(t_1,\ldots,t_{N-1}) := \{\prod_{j=1}^{N-1}\frac{1}{2\pi i}\int_{|\tau_j|=1}\frac{\mu_j(\tau_j)}{\tau_j-t_j}\}\mu_N(\frac{1}{\tau_1\cdots\tau_{N-1}})d\tau_1\cdots d\tau_{N-1}, \tag{5.15}$$

it then follows from (5.7)iii and (5.14) by taking $t_j=0, j=1,\ldots,N-1$,

$$a_{0\cdots0} = I(0,\ldots,0); \tag{5.16}$$

by taking $t_j=0, j=2,\ldots,N-1$,

$$a_{0\cdots0}+a_{10\cdots0}E\{[\mu_1(t_1)]^{\mathbf{k}_{10\cdots0}^{(1)}}|\mathbf{n}_{10\cdots0}<\infty\}=I(t_1,0,\ldots,0); \tag{5.17}$$

by taking $t_j=0, j=3,\ldots,N-1$.

$$\begin{aligned}&a_{0\cdots0}+a_{10\cdots0}E\{[\mu_1(t_1)]^{\mathbf{k}_{10\cdots0}^{(1)}}|\mathbf{n}_{10\cdots0}<\infty\}+\\&\quad a_{010\cdots0}E\{[\mu_2(t_2)]^{\mathbf{k}_{010\cdots0}^{(2)}}|\mathbf{n}_{010\cdots0}<\infty\}+\\&\quad a_{110\cdots0}E\{[\mu_1(t_1)]^{\mathbf{k}_{110\cdots0}^{(1)}}[\mu_2(t_2)]^{\mathbf{k}_{110\cdots0}^{(2)}}|\mathbf{n}_{110\cdots0}<\infty\}=I(t_1,t_2,0,\ldots,0);\end{aligned} \tag{5.18}$$

and so on.

To derive an expression for e.g. $a_{10\cdots0}$ we have to take $t_1\to1$ in (5.17) and then obtain

$$a_{10\cdots0} = I(1,0,\ldots,0)-I(0,0,\ldots,0), \tag{5.19}$$

i.e. we need an expression for

$$I(1,0,\ldots,0) = \lim_{\substack{t_1\to 1\\ |t_1|\leq 1}} I(t,0,\ldots,0). \tag{5.20}$$

By using the Plemelj-Sokhotski formulas for singular integrals, cf. [1], which can be applied because $\mu_j(t_j)$, $|t_j|=1$, satisfies a Hölder condition, cf. [1] and [3], it is seen that for $|t_1|=1$, $|t_j|<1$, $j=2,\ldots,N-1$,

$$I(t_1,t_2,\ldots,t_{N-1}) = \{\prod_{j=1}^{N-1} \frac{1}{2\pi i} \int_{|\tau_j|=1} \frac{\mu_j(\tau_j)}{\tau_j-t_j}\}\mu_N(\frac{1}{\tau_1\cdots\tau_{N-1}})d\tau_1\cdots d\tau_{N-1} \tag{5.21}$$

$$+\tfrac{1}{2}\{\prod_{j=2}^{N-1} \frac{1}{2\pi i} \int_{|\tau_j|=1} \frac{\mu_j(\tau_j)}{\tau_j-t_j}\}\mu_1(t_1)\mu_N(\frac{1}{t_1\tau_2\cdots\tau_{N-1}})d\tau_2\cdots d\tau_{N-1}.$$

By taking in (5.21) $t_1=1$, $t_j=0, j=2,\ldots,N-1$, we obtain the expression for $I(1,0,\ldots,0)$ and find from (5.17) an expression for $a_{00\cdots 0}+a_{10\cdots 0}$. For further details and explicit expressions for the case $N=3$ the reader is referred to [5].

The integral expressions derived above are suitable for numerical evaluation with due care for the singular integrals (see the first integral in (5.21)).

The analysis above (see (5.8), ..., (5.21)) leads to a resolution of the hitting point identity. It is also of interest to express $I(t_1,\ldots,t_N)$ more explicitly in terms of the variables $\mathbf{m}_i$, $i=1,\ldots,N$, see (5.3).

From (5.3) and (5.15) it is seen that for $|t_j|<1, j=1,\ldots,N-1$, we have

$$I(t_1,\ldots,t_{N-1}) = \{\prod_{j=1}^{N-1} \frac{1}{2\pi i} \int_{|\tau_j|=1} \frac{d\tau_j}{\tau_j-t_j}\}E\{\tau_1^{\mathbf{m}_1-\mathbf{m}_N}\cdots\tau_{N-1}^{\mathbf{m}_1-\mathbf{m}_N}\}= \tag{5.22}$$

$$\{\prod_{j=1}^{N-1} \frac{1}{2\pi i} \int_{|\tau_j|=1} \frac{d\tau_j}{\tau_j-t_j}\}E\{\tau_1^{\mathbf{m}_1-\mathbf{m}_N}\cdots\tau_{N-1}^{\mathbf{m}_1-\mathbf{m}_N}[(\mathbf{m}_1-\mathbf{m}_N\geq 0)+(\mathbf{m}_1-\mathbf{m}_N<0)]$$

$$\cdots[(\mathbf{m}_{N-1}-\mathbf{m}_N\geq 0)+(\mathbf{m}_{N-1}-\mathbf{m}_N<0]\}.$$

Because $|t_j|<1$, $j=1,\ldots,N-1$, it is readily verified by contour integrations outside the unit circle that of these 2^{N-1} terms each term which contains for some $j\in\{1,\ldots,N-1\}$ an indicator function $(\mathbf{m}_j-\mathbf{m}_N<0)$ is equal to zero. Consequently, for $|t_j|\leq 1, j=1,\ldots,N-1$,

$$I(t_1,\ldots,t_{N-1}) = E\{t_1^{\mathbf{m}_1-\mathbf{m}_N}\cdots t_{N-1}^{\mathbf{m}_{N-1}-\mathbf{m}_N}(\mathbf{m}_1\geq\mathbf{m}_N)\cdots(\mathbf{m}_{N-1}\geq\mathbf{m}_N)\}; \tag{5.23}$$

that (5.23) also holds for $|t_j|=1$ is a direct result of the continuity in $|t_j\leq 1, j=1,\ldots,N-1$, of both sides of (5.23).

In particular it follows from (5.16), (5.19) and (5.23), (take $t_1=0$ and $t_1=1$),

$$a_{0\cdots 0} = Pr\{\mathbf{m}_1=\mathbf{m}_2=\cdots=\mathbf{m}_N\}. \tag{5.24}$$

$$a_{10\cdots 0} = Pr\{\mathbf{m}_1>\mathbf{m}_2=\cdots=\mathbf{m}_N\}.$$

$$a_{10\cdots 0}E\{[\mu_1(t_1)]^{\mathbf{k}_{10\cdots 0}^{(1)}}|\mathbf{n}_{10\cdots 0}<\infty\}=E\{t_1^{\mathbf{m}_1-\mathbf{m}_N}(\mathbf{m}_1>\mathbf{m}_2=\cdots=\mathbf{m}_N)\}.$$

From the definition of a_b, $\mathbf{x}_b$ and $\mathbf{n}_b$, cf. (2.8), ..., (3.10) the interpretation of the relations (5.24) is obvious.

6. Stochastic vectors associated with ξ.

In this section we shall introduce the concept of a vector η associated with ξ, (actually it is a relation between their distributions).

We start with the case for which ξ has independent components, cf. (5.2), ..., (5.15).

Put

$$S_i := \{p_i: p_i = E\{t_i^{\mathbf{m}_i}\},\ |t_i| = 1\}, \quad i = 1, \ldots, N, \tag{6.1}$$
$$S_i^+ := \{p_i: p_i = E\{t_i^{\mathbf{m}_i}\},\ |t_i| < 1\}, \quad i = 1, \ldots, N,$$

so $\quad [0,1] \subset S_i^+ \cup S_i$.

From the definition of $\mu_i(t_i)$, cf. (5.3), it is readily seen that $\mu_i(t_i)$, $|t_i| \leqslant 1$ is a univalent function, i.e.

$$\mu_i(t) \neq \mu_i(s) \quad \text{for} \quad t \neq s,\ |t| \leqslant 1,\ |s| \leqslant 1. \tag{6.2}$$

Consequently, S_i is a simple contour with S_i^+ its interior, and

$$t_i = \frac{p_i}{\phi_i(p_i)}: S_i^+ \cup S_i \to \{t_i: |t_i| \leqslant 1\}, \tag{6.3}$$

is regular for $p_i \in S_i^+$, continuous for $p_i \in S_i^+ \cup S_i$, and also univalent, hence

$$\phi_i(p_i) \neq 0 \quad \text{for} \quad p_i \in S_i \cup S_i^+.$$

These observations lead directly to

Lemma 6.1. *For $p_i \in S_i^+ \cup S_i$, $i = 1, \ldots, N$,*

$$\sum_{b \in \mathfrak{N}_{b_N=0}} a_b E\{p^{\mathbf{k}_b} | \mathbf{n}_b < \infty\} = \tag{6.4}$$
$$= E\{[\frac{p_1}{\phi_1(p)}]^{\mathbf{m}_1 - \mathbf{m}_N} \cdots [\frac{p_{N-1}}{\phi_{N-1}(p_{N-1})}]^{\mathbf{m}_{N-1} - \mathbf{m}_N} (\mathbf{m}_1 \geqslant \mathbf{m}_N) \cdots (\mathbf{m}_N \geqslant \mathbf{m}_N)\},$$

and

$$\sum_{b \in \mathfrak{N}_{b_j=0}} a_b E\{p^{\mathbf{k}_b} | \mathbf{n}_b < \infty\} = E\{\prod_{k=1}^{N} \left[[\frac{p_k}{\phi_k(p_k)}]^{\mathbf{m}_k - \mathbf{m}_j} (\mathbf{m}_k \geqslant \mathbf{m}_j) \right]\}, \quad j = 1, \ldots, N. \tag{6.5}$$

Proof. The relation (6.4) follows directly from (5.13), (5.15), (5.23) and the fact that $\mu_i(t_i)$, $|t_i| \leqslant 1$, has a unique inverse. The relation (6.5) is a direct consequence of (6.4) and a symmetry argument. □

For the random walk $\mathbf{x}_n$, $n = 0, 1, \ldots$, with structure

$$\mathbf{x}_{n+1} = \mathbf{x}_n - 1 + \boldsymbol{\xi}^{(n)}, \quad n = 0, 1, 2, \ldots, \tag{6.6}$$
$$\mathbf{x}_0 = \boldsymbol{\xi},$$

where $\boldsymbol{\xi}$, $\boldsymbol{\xi}^{(0)}$, $\boldsymbol{\xi}^{(1)}$, ..., is a sequence of i.i.d. stochastic vectors, $\boldsymbol{\xi}^{(n)} \in S$, denote by $\mathbf{k}$ the vector of the hitting point of the boundary B of S at the moment of the first entrance into B.

From the definition of a_b, $\mathbf{k}_b$ and $\mathbf{n}_b$, cf. (2.10), it follows directly that the N-variate generating function of $\mathbf{k}$ is given by

$$E\{p^{\mathbf{k}}\} = \sum_{b \in \mathfrak{N}} a_b E\{p^{\mathbf{k}_b} | \mathbf{n}_b < \infty\}, \quad |p| \leqslant 1; \tag{6.7}$$

and for the case that $\boldsymbol{\xi}$ has independent components (and assumption 3.1 applies) this N-variate generating function can be determined from (6.5).

From the hitting point identity, cf. (4.2), it follows that every zero tuple $\hat{p}$ of

$$p^1 - E\{p^{\boldsymbol{\xi}}\}, \quad |p| \leqslant 1,$$

is also a zero tuple of the kernel of $\mathbf{k}$, i.e. of

$$p^1 - E\{p^{\mathbf{k}}\}, \quad |p| \leqslant 1.$$

DEFINITION 6.1. A stochastic vector η is said to be *associated* with the stochastic vector ξ if every zero tuple $\hat{p}$ of

$$p^1 - E\{p^{\xi}\}, \quad |p| \geqslant 1,$$

is also a zero tuple of

$$p^1 - E\{p^{\eta}\}, \quad |p| \geqslant 1.$$

Obviously, the stochastic variable $\mathbf{k}$ with N-variate generating function given by (6.7) is associated with ξ.

REMARK 6.1. The concept of 'associated' is actually a relation between the distributions of ξ and η; however, we rather prefer the phrasing with stochastic variables than with distributions.

Denote by $A(\xi)$ the class of distributions of stochastic variables associated with ξ.

Actually $A(\xi)$ is a very large class, and we briefly discuss some techniques to construct elements of $A(\xi)$ (it will not be assumed that ξ has independent components).

i. $A(\xi)$ is a convex class. Evidently, if η_1 and η_2 belong to $A(\xi)$ then

$$c_1 E\{p^{\eta_1}\} + c_2 E\{p^{\eta_2}\}, \ |p| \leqslant 1, \ 1 \geqslant c_1 \geqslant 0, \ 1 \geqslant c_2 \geqslant 0, \ c_1 + c_2 = 1,$$

is an N-variate generating function of a distribution with support contained in S, and for any zero tuple $\hat{p}$ of $p^1 - E\{p^{\xi}\}$ we have

$$c_1 E\{p^{\eta_1}\} + c_2 E\{p^{\eta_2}\} = c_1 \hat{p}^1 + c_2 \hat{p}^1 = \hat{p}^1;$$

so that η with distribution the convex combination of the distributions of η_1 and η_2 is associated with ξ.

ii. Let $D \subset S \setminus B$ and such that $S \setminus D$ is a connected set, i.e. any two points in $S \setminus D$ can be connected by a path along neighbouring points all belonging to $S \setminus D$; two points of S are each others neighbours if all their corresponding coordinates except one are equal and the corresponding coordinates which are unequal differ by one (in absolute value). Consider the random walk $\mathbf{x}_n$, $n = 0, 1, 2, \ldots$, defined by: for $n = 0, 1, \ldots$

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n - 1 + \xi^{(n)} && \text{if } \mathbf{x}_n \in S \setminus \{D \cup B\}, \\ &= \mathbf{x}_n && \text{if } \mathbf{x}_n \in D \cup B, \\ \mathbf{x}_0 &= \xi, \end{aligned}$$

with $\xi, \xi^{(0)}, \xi^{(1)}, \ldots$, i.i.d. vectors. Hence $D \cup B$ is an absorbing set for this random walk.

Denote by $\mathbf{k}$ the hitting point of $D \cup B$ at the first entrance into this set, by $\mathbf{n}_D$ the first entrance time into D and by $\mathbf{k}_D$ the hitting point of D, then for $|p| \leqslant 1$,

$$E\{p^{\mathbf{k}}\} = E\{p^{\mathbf{k}_D}(\mathbf{n}_D < \infty)\} + \sum_{b \in \mathfrak{N}} E\{p^{\mathbf{k}_b}(\mathbf{n}_b < \infty)\}, \tag{6.8}$$

with $\mathbf{n}_b$ the first entrance time in $B(b)$ and $\mathbf{k}_b$ the hitting point of $B(b)$ (the distributions of $\mathbf{k}_b$ and $\mathbf{n}_b$ depend on D).

As above it may be shown that $\mathbf{k}$ is associated with ξ. For details we refer the reader to [5], in particular for the resolution of the identity connected with (6.8), i.e.

$$\hat{p}^1 = E\{\hat{p}^{\mathbf{k}}\} = E\{\hat{p}^{\mathbf{k}_D}(\mathbf{n}_D < \infty)\} + \sum_{b \in \mathfrak{N}} E\{\hat{p}^{\mathbf{k}_b}(\mathbf{n}_b < \infty)\}, \tag{6.9}$$

with $\hat{p}$ a zero tuple of

$$p^1 - E\{p^{\xi}\}, \quad |p| \leqslant 1.$$

iii. The distribution of the generating function (6.8) has obviously as support the set $D \cup B$, note that the right-hand side of (6.9) does not contain a term in which all the p_j, $j = 1, \ldots, N$ occur. It is, however, easy to construct for every $M \geqslant 2$ a distribution with support $\{0,1,2,\ldots\}^M$. We shall indicate this for $M < N$, for $M > N$ the construction will then become evident.
Obviously, $\hat{p}$ with

$$\begin{aligned} \hat{p}_i &= E\{t_i^{\mathbf{m}_i}\}, \quad |t_i| = 1, \quad i = 1, \ldots, M; \quad \prod_{i=1}^{M} t_i = 1. \\ &= 1, \qquad\qquad\qquad i = M+1, \ldots, N, \end{aligned} \tag{6.10}$$

is a zero tuple of

$$\prod_{i=1}^{M} p_i - \prod_{i=1}^{M} \phi_i(p_i), \quad |p_i| \leqslant 1,$$

and hence also of cf. (6.8),

$$\prod_{i=1}^{M} p_i - E\{p^{\mathbf{k}}\}, \quad |p| \leqslant 1, \quad p_i = 1, \quad i = M+1, \ldots, N.$$

Hence a stochastic variable $\mathbf{k}_M$ with distribution the projection of the distribution of $\mathbf{k}$ onto $\{0,1,2,\ldots\}^M$ is associated with the stochastic vector $\{\xi_1, \ldots, \xi_M\}$. It should be noted that the distribution of $\mathbf{k}_M$ depends on $\phi_{M+1}(\cdot), \ldots, \phi_N(\cdot)$.

7. A relation between moments

Let ξ and η both be stochastic variable with state space $\{0,1,2,\ldots\}^N$.

$$\begin{aligned} \xi &= \{\xi_1, \ldots, \xi_N\}, \quad & \eta &= \{\eta_1, \ldots, \eta_N\}, \\ \phi(p) &:= E\{p^{\xi}\}, & \omega(p) &:= E\{p^{\eta}\}, \qquad |p| \leqslant 1, \end{aligned} \tag{7.1}$$

with ξ satisfying ass. (3.11) and $\omega(p)$ given by, cf. (6.5),

$$\omega(p) = \sum_{b \in \mathfrak{N}} a_b E\{p^{\mathbf{k}_b} | \mathbf{n}_b < \infty\}, \quad |p| \leqslant 1. \tag{7.2}$$

Hence, cf. the discussion following (6.7), it is seen that every zero tuple $\hat{p}$ of

$$p^1 - E\{p^{\xi}\}, \quad |p| \leqslant 1, \tag{7.3}$$

is a zero tuple of

$$p^1 - E\{p^{\eta}\}, \quad |p| \leqslant 1. \tag{7.4}$$

Theorem 7.1. *For $j = 1, \ldots, N$,*

$$\frac{E\{1-\eta_j\}}{E\{1-\xi_j\}} \quad \textit{is independent of } j. \tag{7.5}$$

Proof. From (3.11)i and ii it follows by applying Rouché's theorem that the function

$$p_1 p_2 - E\{p_1^{\xi_1} p_2^{\xi_2}\}, \quad |p_1| \leqslant 1, \quad |p_2| \leqslant 1, \tag{7.6}$$

has for $|p_2| = 1$ exactly one zero $p_1(p_2)$ in $|p_1| \leqslant 1$, that

$$\left|\frac{dp_1(p_2)}{dp_2}\right| < \infty, \quad |p_2| \leqslant 1, \tag{7.7}$$

and

$$-\frac{dp_1(p_2)}{dp_2}\Big|_{p_2=1} = \frac{1-E\{\xi_2\}}{1-E\{\xi_1\}}. \tag{7.8}$$

Obviously

$$\hat{p} = (\hat{p}_1, \ldots, \hat{p}_N), \tag{7.9}$$

with

$$\hat{p}_1 = p_1(\hat{p}_2), \ |\hat{p}_2|=1, \ \hat{p}_j=1, \ j=3,\ldots,N, \tag{7.10}$$

is a zero tuple of (7.3) and hence also of (7.4). Consequently

$$E\{\hat{p}^{\xi}\} = E\{\hat{p}^{\eta}\}=\hat{p}^{1}. \tag{7.11}$$

From the first relation of (7.11) it follows that

$$-\frac{\hat{p}_2}{\hat{p}_1}\frac{d\hat{p}_1}{d\hat{p}_2} = \frac{E\{\xi_2\hat{p}_1^{\xi_1}\hat{p}_2^{\xi_2}\}-E\{\eta_2\hat{p}_1^{\eta_1}\hat{p}_2^{\eta_2}\}}{E\{\xi_1\hat{p}_1^{\xi_1}\hat{p}_2^{\xi_2}\}-E\{\eta_1\hat{p}_1^{\eta_1}\hat{p}_2^{\eta_2}\}}, \ |\hat{p}_2|=1, \tag{7.12}$$

and from the second relation of (7.11),

$$-\frac{\hat{p}_2}{p_1}\frac{d\hat{p}_1}{d\hat{p}_2} = \frac{E\{(1-\eta_2)\hat{p}_1^{\eta_1}\hat{p}_2^{\eta_2}\}}{E\{(1-\eta_2)\hat{p}_1^{\eta_1}\hat{p}_2^{\eta_2}\}}, \ |\hat{p}_2|=1. \tag{7.13}$$

By letting $\hat{p}_2 \to 1$ so that, cf. (7.6), $\hat{p}_1 \to 1$ it follows from (7.7), (7.8), (7.12) and (7.13) that

$$\frac{1-E\{\xi_2\}}{1-E\{\xi_1\}} = \frac{E\{\xi_2\}-E\{\eta_2\}}{E\{\xi_1\}-E\{\eta_1\}} = \frac{1-E\{\eta_2\}}{1-E\{\eta_1\}} = -\frac{d\hat{p}_1}{d\hat{p}_2}\Big|_{\hat{p}_2=1}, \tag{7.14}$$

or

$$E\{\eta_1\} = E\{\xi_1\} \quad \text{and} \quad E\{\eta_2\} = E\{\xi_2\}, \tag{7.15}$$

or

$$E\{\eta_1\} = E\{\eta_2\} = 1. \tag{7.16}$$

Note that the three linear equations between the four moments in (7.14) are dependent. Obviously the relation (7.14) between these four moments holds for any pair $i,j \subset \{1,\ldots,N\}$; the relation (7.5) has been proved.

THEOREM 7.2. *It ξ has independent components then for $j=1,2,\ldots,N$,*

$$0<E\{\eta_j\} < 1. \tag{7.17}$$

PROOF. From (5.2), (6.5) and (7.2) it follows for $p_k \in S_k^+ \cup S_k$, by differentiating with respect to p_1 and taking $p_j=1$, $j=1,\ldots,N$ that

$$E\{\eta_1\}=[1-E\{\xi_1\}]\sum_{j=1}^{N} E\{(\mathbf{m}_1-\mathbf{m}_j)\prod_{k=1}^{N}(\mathbf{m}_k \geq \mathbf{m}_j)\}. \tag{7.18}$$

Since, cf. (5.4),

$$1 \leq E\{\mathbf{m}_j\} = \frac{1}{1-E\{\xi_1\}} < \infty, \tag{7.19}$$

it is seen that

$$0<E\{\eta_1\}<[1-E\{\xi_1\}]E\{\mathrm{m}_1\sum_{j=2}^{N}\prod_{k=1}^{N}(\mathrm{m}_k\geq\mathrm{m}_j)\}<[1-E\{\xi_1\}]E\{\mathrm{m}_1\}=1,$$

analogously for $E\{\eta_j\}, j=2,\ldots,N$. □

REMARK 7.1. Theorems 7.1 and 7.2 are actually weaker than the statement ii. of theorem 4.3. However, with a view of the concept of associated stochastic variables, cf. section 9, it is desirable to present another proof of (7.5).

THEOREM 7.3. *Let ξ and η be stochastic variables both satisfying (3.11) with ξ having independent components, and let $\{\mathrm{x}_n, n=0,1,\ldots\}$ and $\{\mathrm{y}_n, n=0,1,\ldots\}$ both be absorbing random walks defined by: for $n=0,1,2,\ldots,$*

$$\begin{aligned} \mathrm{x}_{n+1} &= \mathrm{x}_n-1+\xi_n, & \mathrm{x}_n&\in S\setminus B, & \mathrm{y}_{n+1} &= \mathrm{y}_n-1+\eta_n, & \mathrm{y}_n&\in S\setminus B,\\ &= \mathrm{x}_n, & \mathrm{x}_n&\in B, & &= \mathrm{y}_n, & \mathrm{y}_n&\in B,\\ \mathrm{x}_0 &= x_0, & x_0&\in S, & \mathrm{y}_0 &= y_0, & y_0&\in B, \end{aligned} \tag{7.20}$$

with $\{\xi_n, n=0,1,\ldots\}$ and $\{\eta_n, n=0,1,\ldots\}$ both sequences of i.i.d. stochastic variables with distribution that of ξ and η, respectively; then these random walks have identical hitting point distributions of the boundary B for $x_0=y_0$, if η is associated with ξ.

PROOF. Since η is associated with $\overline{\xi}$, any zero tuple $\hat{p}$ of

$$p^1-E\{p^{\xi}\},\quad |p|\leq 1, \tag{7.21}$$

is a zero tuple of

$$p^1-E\{p^{\eta}\},\quad |p|\leq 1, \tag{7.22}$$

(see def. 6.1). Since ξ has independent components, we take for $\hat{p}$ the zero tuple given by (5.3)

$$\hat{p}_j = E\{t_j^{\mathrm{m}_j}\},\quad |t_j|=1,\quad \prod_{j=1}^{N}t_j=1,\quad j=1,\ldots,N. \tag{7.23}$$

Denote by $\mathrm{k}_x(x_0)$ and $\mathrm{k}_y(y_0)$ the hitting point of B for the random walk x_n and y_n, respectively, cf. (2.8). Since (3.11) holds for ξ as well as η it follows from Theorem 4.2 with $\hat{p}$ the zero tuple (7.23) that

$$\hat{p}^{x_0} = E\{\hat{p}^{\mathrm{k}_x(x_0)}\}=\sum_{b\in\mathfrak{N}}a_{xb}(x_0)E\{\hat{p}^{\mathrm{k}_{xb}(x_0)}|\mathrm{n}_{xb}(x_0)<\infty\}, \tag{7.24}$$

$$\hat{p}^{y_0} = E\{\hat{p}^{\mathrm{k}_y(y_0)}\}=\sum_{b\in\mathfrak{N}}a_{yb}(y_0)E\{\hat{p}^{\mathrm{k}_{yb}(y_0)}|\mathrm{n}_{xb}(y_0)<\infty\}, \tag{7.25}$$

where the symbols referring to the x_n and the y_n random walk have been indexed by x and y, respectively.

By applying the integral operator, cf. (5.8),

$$\{\prod_{j=1}^{N}\frac{1}{2\pi i}\int_{|\tau_j|=1}\frac{d\tau_j}{\tau_j-t_j}\}\cdots,\quad |t_j|<1,\quad j=1,\ldots,N, \tag{7.26}$$

for every term of the sum in the right-hand side of (7.24) an explicit expression can be obtained, cf. the discussion in Section 5, see e.g. (5.17). Similarly, for the terms of the sum in the right-hand side of (7.25). Obviously for $x_0=y_0$ the expressions for corresponding terms of these sums are identical. Once these terms are known the generating function of the distribution of $\mathrm{k}_x(x_0)$ and that of $\mathrm{k}_y(y_0)$ may be found by using the inverse mapping (6.3), leading to a relation similar to (6.7). Since the corresponding terms just mentioned are equal for $x_0=y_0$, the generating functions are identical for $x_0=y_0$ and the theorem has been proved. □

REMARK 7.2. The proof of the theorem above is based on the independence of the components of ξ. A proof of this theorem without the independence assumption can be constructed from the results in [3].

8. THE REFLECTING RANDOM WALK

In this section we consider the random walk $\{\mathbf{x}_n, n=0,1,2,\ldots\}$ on S with nonabsorbing boundary B, i.e. in multi-index notation: for $n=0,1,2,\ldots,$

$$\begin{aligned} \mathbf{x}_{n+1} &= [\mathbf{x}_n-1]^+ + \xi^{(n)}, \qquad (8.1)\\ \mathbf{x}_0 &= x_0, \end{aligned}$$

with $\xi^{(n)}$, $n=0,1,\ldots,$ i.i.d. stochastic vectors, cf. (2.1) and (3.11), so that its state space S is irreducible.

Put for $n=0,1,2,\ldots;\ |p|\leqslant 1,$

$$\Phi_{x_0}^{(n)}(p) := E\{p^{\mathbf{x}_n} | \mathbf{x}_0 = x_0\}. \tag{8.2}$$

Since $\mathbf{x}_n$ and $\xi^{(n)}$ are independent vectors it follows from (8.1) and (8.2) that: for $n=0,1,2,\ldots,$ $|p|\leqslant 1,$

$$\begin{aligned} \Phi_{x_0}^{(n+1)}(p) &= E_{x_0}\{\prod_{m=1}^{N} p_m^{[\mathbf{x}_n^{(m)}-1]^+ + \xi_m^{(m)}}\} = E\{p^{\xi}\}E_{x_0}\{\prod_{m=1}^{N}[p_m^{\mathbf{x}_n^{(m)}-1}(\mathbf{x}_n^{(m)}>0)+(\mathbf{x}_n^{(m)}=0)]\} \qquad (8.3)\\ &= \frac{E\{p^{\xi}\}}{p^1}E_{x_0}\{\prod_{m=1}^{N}[p_m^{\mathbf{x}_n^{(m)}} + (p_m-1)(\mathbf{x}_n^{(m)}=0)]\}. \end{aligned}$$

Put for $|r|<1$, $|p|\leqslant 1,$

$$\Phi_{x_0}(r,p) := \sum_{n=0}^{\infty} r^n \Phi_{x_0}^{(n)}(p), \tag{8.4}$$

it then follows from (8.3) and (8.4): for $|r|<1$, $|p|\leqslant 1,$

$$[p^1 - rE\{p^{\xi}\}]\Phi_{x_0}(r,p) = p^{x_0+1} + rE\{p^{\xi}\}\sum_{b\in\mathfrak{N}}(p(1-b)-1)^1\Phi_{x_0}(r,pb), \tag{8.5}$$

with for $|p|\leqslant 1$, $b\in\mathfrak{N}$,

$$pb = (p_1b,\ldots,p_Nb_N),\ (p(1-b)-1)^1 = \prod_{k=1}^{N}\{p_k(1-b_k)-1\}. \tag{8.6}$$

The relation (8.5) represents the functional equation for the function $\Phi_{x_0}(r,p)$.

Our interest here concerns the stationary distribution of the process $\mathbf{x}_n$; it exists because of ass. (3.11)i, as it follows directly from Theorem 4.1.iii. and because (3.11)iii implies that its state space S is irreducible. For the transient case see [3].

Let

$$\mathbf{x} = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)\in S,$$

be a stochastic variable with distribution this stationary distribution and put for $|p|\leqslant 1,$

$$\Phi(p) := E\{p^{\mathbf{x}}\}. \tag{8.7}$$

It is well-known that for $|p|\leqslant 1,$

$$\Phi(p) = \lim_{r\uparrow 1}(1-r)\Phi_{x_0}(r,p). \tag{8.8}$$

Hence it follows from (8.5) for $|p|\leqslant 1,$

$$[p^1 - E\{p^{\xi}\}]\Phi(p) = (p-1)^1 E\{p^{\xi}\}\sum_{b\in\mathfrak{N}}\frac{\Phi(pb)}{(pb-1)^1}. \tag{8.9}$$

The relation (8.9) represents the functional equation for $\Phi(p)$, $|p|\leqslant 1$, and with this relation we can now formulate the conditions which $\Phi(p)$, $|p|\leqslant 1$ has to satisfy.

THEOREM 8.1. *The function $\Phi(p)$, $|p|\leqslant 1$, cf. (8.7), satisfies the following conditions:*

i. $\Phi(p) = 1$ *for* $p=1$, *i.e.* $p_j=1, j=1,\dots,N$; (8.10)

ii. *for every $j\in\{1,2,\dots,N\}$ the function $\Phi(p)$ is regular in p_j for $|p_j|<1$, continuous in p_j for $|p_j|\leqslant 1$, the other variables p_k, $k\neq j$, kept fixed with $|p_k|\leqslant 1$;*

iii. *for* $|p|\leqslant 1$,

$$[p^1-E\{p^{\xi}\}]\Phi(p) = (p-1)^1 E\{p^{\xi}\}\sum_{b\in\mathfrak{N}}\frac{\Phi(pb)}{(pb-1)^1}.$$

iv. $$\frac{(\hat{p}-1)^1}{\hat{p}^1-1}\sum_{b\in\mathfrak{N}}\frac{\Phi(\hat{p}b)}{(\hat{p}b-1)^1} = 0, \text{ for every zero tuple } \hat{p} \text{ of the kernel}$$

$$p^1-E\{p^{\xi}\},\quad |p|\leqslant 1.$$

PROOF. Above it has been argued that the process x_n possesses a stationary distribution; since $\Phi(p)$ is the N-variate generating function it is seen that (8.10)i formulates the norming condition. The condition (8.10)ii follows directly from the definition (8.7), whereas (8.10)iii has been derived above, cf. (8.9). Because (8.7) implies that $|\Phi(p)|<\infty$ for $|p|\leqslant 1$ the relation (8.10)iv follows directly from Remark 3.1 (viz. $\phi(p)\neq 0$ if $p^1=0$) and from (8.9) for $\hat{p}^1\neq 1$; and also for $\hat{p}^1\to 1$ since $\hat{p}=1$ is a common zero of p^1-1 and the kernel. □

THEOREM 8.2. *If ξ has independent components, cf. (5.2), then*

i. $$\Phi(p) = \prod_{k=1}^{N}\{[1-E\{\xi_k\}]\frac{(1-p_k)\phi_k(p_k)}{\phi_k(p_k)-p_k}\},\quad |p|\leqslant 1, \tag{8.11}$$

and

ii. $\Phi(p)$ *is uniquely determined by the conditions (8.10)i, ii and iv.*

REMARK 8.1. If ξ has independent components then the components $x_n^{(j)}$, $j=1,\dots,N$, of the process x_n are obviously independent and since $E\{\xi_j\}<1$ each component process possesses a stationary distribution of which the generating function is given by

$$[1-E\{\xi_j\}]\frac{(1-p_j)\phi_j(p_j)}{\phi_j(p_j)-p_j},\quad |p_j|\leqslant 1,\quad j=1,\dots,N,$$

and hence (8.11)i follows.

PROOF OF THEOREM 8.2. It is readily verified that $\Phi(p)$ as given by (8.11)i satisfies the conditions (8.10).

To prove (8.11)ii we first prove it for $N=2$ and then for $N=3$, the construction of the proof for general N will then be evident.

Proof of (8.11)ii for $N=2$.

Since ξ has independent components

$$\hat{p}_1 = \mu_1(t)=E\{t^{m_1}\},\ \hat{p}_2=\mu_2(\frac{1}{t})=E\{t^{-m_1}\},\quad |t|=1, \tag{8.12}$$

is a zero tuple of the kernel (cf. (5.2), . . . , (5.3)),

$$p_1p_2 - E\{p_1^{\xi_1}\}E\{p_2^{\xi_2}\}, \quad |p_1| \leqslant 1, \quad |p_2| \leqslant 1, \tag{8.13}$$

and so the condition (8.10)iv reads

$$\frac{(\hat{p}_1-1)(\hat{p}_2-1)}{\hat{p}_1\hat{p}_2-1}\{\Phi(0,0)+\frac{\Phi(\hat{p}_1,0)}{\hat{p}_1-1}+\frac{\Phi(0,\hat{p}_2)}{\hat{p}_2-1}\} = 0, \tag{8.14}$$

or equivalently, cf. (8.12),

$$(t-1)[\Phi(0,0)+\frac{\Phi(\mu_1(t),0)}{\mu_1(t)-1}+\frac{\Phi(0,\mu_2(\frac{1}{t}))}{\mu_2(\frac{1}{t})-1}]=0, \quad |t|=1. \tag{8.15}$$

Rewrite (8.15) as

$$(t-1)\Phi(0,0)+\frac{t-1}{\mu_1(t)-1}\Phi(\mu_1(t),0) = -\frac{t-1}{\mu_2(\frac{1}{t})-1}\Phi(0,\mu_2(\frac{1}{t})), \quad |t|=1. \tag{8.16}$$

Since

$$E\{\xi_j\}<1, \quad j=1,2,$$

$\mu_j(t)-1$ has a zero of multiplicity one at $t=1$. Further $\mu_1(t)-1$ is regular for $|t|<1$, continuous and nonzero for $|t|\leqslant 1$, $t\neq 1$, cf. also (3.11)ii; similarly, for $\mu_2(\frac{1}{t})-1$ with $|t|\geqslant 1$. The condition (8.11)ii implies that $\Phi(\mu_1(t),0)$ is regular for $|t|<1$, continuous for $|t|\leqslant 1$, and similarly, for $\Phi(0,\mu_2(\frac{1}{t}))$ with $|t|\geqslant 1$. Hence the left-hand side of (8.16) for $|t|\leqslant 1$ and the right-hand side of (8.16) for $|t|\geqslant 1$ are each other analytic continuation. So that, since

$$\mu_2(\frac{1}{t})\to 0 \quad \text{for} \quad |t|\to\infty,$$

Liouville's theorem implies that

$$(t-1)\Phi(0,0)+\frac{t-1}{\mu_1(t)-1}\Phi(\mu_1(t),0) = (t-1)C_1+C_2, \quad |t|\leqslant 1, \tag{8.17}$$

$$-\frac{t-1}{\mu_1(t)-1}\Phi(0,\mu_2(\frac{1}{t})) = (t-1)C_1+C_2, \quad |t|\geqslant 1,$$

with C_1 and C_2 constants, i.e. independent of t. For $|t|\to\infty$ it follows from (8.17) that

$$C_1 = \Phi(0,0); \tag{8.18}$$

and with $t=0$ from (8.16),

$$C_1 = C_2; \tag{8.19}$$

so

$$\Phi(\mu_1(t),0) = \frac{\mu_1(t)-1}{t-1}\Phi(0,0), \quad |t|\leqslant 1, \tag{8.20}$$

$$\Phi(0,\mu_1(\frac{1}{t})) = \frac{\mu_2(\frac{1}{t})-1}{\frac{1}{t}-1}\Phi(0,0), \quad |t|\geqslant 1.$$

Hence, cf. (6.3),

$$\Phi(p_1,0) = \frac{(1-p_1)\phi_1(p_1)}{\phi_1(p_1)-p_1}\Phi(0,0), \quad p_1 \in S_1^+ \cup S_1, \tag{8.21}$$

$$\Phi(0,p_2) = \frac{(1-p_2)\phi_2(p_2)}{\phi_2(p_2)-p_2}\Phi(0,0), \quad p_2 \in S_2^+ \cup S_2.$$

Because the right-hand sides of (8.21) are regular for $|p_1|<1$, continuous for $|p_1|\leqslant 1$, respectively with p_1 replaced by p_2, the relations (8.21) hold by analytic continuation for $|p_1|\leqslant 1$ and $|p_2|\leqslant 1$, respectively. From (8.10)iii and (8.21) it follows that for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$\Phi(p_1,p_2) = \frac{(1-p_1)\phi_1(p_1)}{\phi_1(p_1)-p_1}\,\frac{(1-p_2)\phi_2(p_2)}{\phi_2(p_2)-p_2}\Phi(0,0), \tag{8.22}$$

and the norming condition yields

$$\Phi(0,0) = [1-E\{\xi_1\}][1-E\{\xi_2\}],$$

and consequently it follows that for $N=2$ the conditions (8.10)i, ii and iv determine $\Phi(p)$, $|p|\leqslant 1$ uniquely. □

Proof of Theorem 8.2 for $N=3$. Again by using the fact that ξ has independent components we take as a zero tuple of the kernel, cf. (5.1), . . . , (5.3),

$$\hat{p}_1 = \mu_1(t_1), \quad \hat{p}_2 = \mu_2(t_2), \quad \hat{p}_3 = \mu_3(\overline{\frac{1}{t_1t_2}}), \quad |t_1|=1, \quad |t_2|=1. \tag{8.23}$$

The condition (8.10)iv is now equivalent with: for $|t_1|=1$, $|t_2|=1$,

$$(t_1-1)(t_2-1)[\Phi(0,0,0)+\frac{\Phi(\hat{p}_1,0,0)}{\hat{p}_1-1}+\frac{\Phi(0,\hat{p}_2,0)}{\hat{p}_2-1}+\frac{\Phi(0,0,\hat{p}_3)}{\hat{p}_3-1}+ \tag{8.24}$$

$$\frac{\Phi(\hat{p}_1,\hat{p}_2,0)}{(\hat{p}_1-1)(\hat{p}_2-1)}+\frac{\Phi(\hat{p}_1,0,\hat{p}_3)}{(\hat{p}_1-1)(\hat{p}_3-1)}+\frac{\Phi(0,\hat{p}_2,\hat{p}_3)}{(\hat{p}_2-1)(\hat{p}_3-1)}] = 0.$$

By letting $t_1 \to 1$ we obtain from (8.24) for $|t_2|=1$,

$$\frac{1}{E\{\mathbf{m}_1\}}[(t_2-1)\Phi(1,0,0)+\frac{t_2-1}{\mu_2(t_2)-1}\Phi(1,\mu_2(t_2),0)+\frac{t_2-1}{\mu_3(\frac{1}{t_2})-1}\Phi(1,0,\mu_2(\frac{1}{t_2}))] = 0. \tag{8.25}$$

The relation (8.25) has exactly the some structure as (8.15) and its unique solution reads, cf. (8.20),

$$\Phi(1,\mu_2(t_2),0) = \frac{\mu_2(t_2)-1}{t_2-1}\Phi(1,0,0), \quad |t_2|\leqslant 1, \tag{8.26}$$

$$\Phi(1,0,\mu_3(\frac{1}{t_2})) = \frac{\mu_3(\frac{1}{t_2})-1}{\frac{1}{t_2}-1}\Phi(1,0,0), \quad |t_2|\geqslant 1.$$

Hence, cf. the derivation of (8.21),

$$\Phi(1,p_2,0) = \frac{(1-p_2)\phi_2(p_2)}{\phi_2(p_2)-p_2}\Phi(1,0,0), \quad |p_2|\leqslant 1, \tag{8.27}$$

$$\Phi(1,0,p_3) = \frac{(1-p_3)\phi_3(p_3)}{\phi_3(p_3)-p_3}\Phi(1,0,0), \quad |p_3|\leqslant 1,$$

and the expressions for

$$\Phi(p_1,1,0),\ \Phi(0,1,p_3),\ \Phi(p_1,0,1),\ \Phi(0,p_2,1), \tag{8.28}$$

can be obtained by using the symmetry. It remains to determine $\Phi(p_1,p_2,0)$, once this function has been constructed the determination of $\Phi(p_1,0,p_3)$ and $\Phi(0,p_2,p_3)$ follows by symmetry.

Rewrite (8.24) as: for $|t_1|=1$, $|t_2|=1$,

$$\begin{aligned}(t_1-1)(t_2-1)\frac{\Phi(\hat p_1,\hat p_2,0)}{(\hat p_1-1)(\hat p_2-1)} &= -(t_1-1)(t_2-1)[\frac{\Phi(0,0,\hat p_3)}{\hat p_3-1}+\Phi(0,0,0)]+\\ &-(t_1-1)(t_2-1)[\frac{\Phi(\hat p_1,0,\hat p_3)}{(\hat p_1-1)(\hat p_3-1)}+\frac{\Phi(\hat p_1,0,0)}{\hat p_1-1}]+\\ &-(t_1-1)(t_2-1)[\frac{\Phi(0,\hat p_2,\hat p_3)}{(\hat p_2-1)(\hat p_3-1)}+\frac{\Phi(0,\hat p_2,0)}{\hat p_2-1}].\end{aligned} \tag{8.29}$$

We note that for fixed $|t_1|=1$, the function (cf. (8.23))

$$\begin{aligned}&(t_1-1)(t_2-1)[\frac{\Phi(\hat p_1,0,\hat p_3)}{(\hat p_1-1)(\hat p_3-1)}+\frac{\Phi(\hat p_1,0,0)}{\hat p_1-1}]\\ &=(t_1-1)(t_2-1)[\frac{\Phi(\mu_1(t_1),0,\mu_3(\frac{1}{t_1t_2}))}{\{\mu_1(t_1)-1\}\{\mu_3(\frac{1}{t_1t_2})-1\}}+\frac{\Phi(\mu_1(t_1),0,0)}{\mu_1(t_1)-1}]\end{aligned} \tag{8.30}$$

is continuous for $|t_2|\geqslant 1$, regular for $\infty\geqslant|t_2|>1$. By using the notation

$$\Phi_3^{(k)}(p_1,p_2,p_3):=\frac{d^k}{dp_3^k}\Phi(p_1,p_2,p_3),\quad k=1,2,\ldots, \tag{8.31}$$

and by noting that $\Phi(\cdot,\cdot,p_3)$ is regular in $|p_3|<1$, it is seen that for fixed $|t_1|=1$,

$$\begin{aligned}&(t_1-1)(t_2-1)[\frac{\Phi(\mu_1(t_1),0,\mu_3(\frac{1}{t_1t_2}))}{\{\mu_1(t_1)-1\}\{\mu_3(\frac{1}{t_1t_2})-1\}}+\frac{\Phi(\mu_1(t_1),0,0)}{\mu_1(t_1)-1}]\\ &=\frac{t_1-1}{\mu_1(t_1)-1}[(t_2-1)\frac{\Phi(\mu_1(t_1),0,0)}{\mu_3(\frac{1}{t_1t_2})-1}+\Phi(\mu_2(t_1),0,0)\}+\\ &\frac{(t_2-1)\mu_3(\frac{1}{t_1t_2})}{\mu_3(\frac{1}{t_1t_2})-1}\{\Phi_3^{(1)}(\mu_1(t_1),0,0)+o(\frac{1}{t_1t_2})\}]\quad\text{for}\quad |t_2|\to\infty.\end{aligned} \tag{8.32}$$

Consequently, the function in the left-hand side of (8.30) has a first order pole at $|t_2|=\infty$ for fixed $|t_1|=1$.

We now apply the operator

$$\left[\frac{1}{2\pi i}\right]^2\int\limits_{|\tau_1|=1}\int\limits_{|\tau_2|=1}\frac{d\tau_1}{\tau_1-t_1}\frac{d\tau_2}{\tau_2-t_2}\cdots\quad\text{with}\quad|t_1|<1,\quad|t_2|<1, \tag{8.33}$$

to the equation (8.39).

Because of (8.10)ii and (8.23) we have by applying Cauchy's theorem: for $|t_1|<1$, $|t_2|<1$,

$$\left[\frac{1}{2\pi i}\right]^2 \int_{|\tau_1|=1}\int_{|\tau_2|=1} \frac{d\tau_1}{\tau_1-t_1}\,\frac{d\tau_2}{\tau_2-t_2}(t_1-1)(t_2-1)\frac{\Phi(\mu_1(\tau_1),\mu_2(\tau_2),0)}{\{\mu_1(\tau_1)-1\}\{\mu_2(\tau_2)-1\}} = \tag{8.34}$$
$$(t_1-1)(t_2-1)\frac{\Phi(\mu_1(t_1),\mu_2(t_2),0)}{\{\mu_1(t_1)-1\}\{\mu_2(t_2)-1\}}\,.$$

To evaluate the operator (8.33) applied to the right-hand side of (8.29) we first consider the application of the operator (8.33) to the expression (8.32). So for $|t_1|<1$, $|t_2|<1$ and R_2 sufficiently large

$$\left[\frac{1}{2\pi i}\right]^2 \int_{|\tau_1|=1}\int_{|\tau_2|=1} \frac{\tau_1-1}{\tau_1-t_1}\,\frac{\tau_2-1}{\tau_2-t_2}\left[\frac{\Phi(\mu_1(\tau_1),0,\mu_3(\frac{1}{\tau_1\tau_2}))}{\{\mu_1(\tau_1)-1\}\{\mu_3(\frac{1}{\tau_1\tau_2})-1\}}+\frac{\Phi(\mu_1(\tau_1),0,0)}{\mu_1(\tau_1)-1}\right]d\tau_1,d\tau_2 \tag{8.35}$$
$$=\frac{1}{2\pi i}\frac{(\tau_1-1)d\tau_1}{(\tau_1-t_1)(\mu_1(\tau_1)-1)}\Big[$$
$$\frac{1}{2\pi i}\int_{|\tau_2|=R_2}\frac{\tau_2-1}{\tau_2-t_2}\,\frac{\mu_3(\frac{1}{\tau_1\tau_2})}{\mu_3(\frac{1}{\tau_1\tau_2})-1}\{\Phi_1(\mu_1(\tau_1),0,0)+\Phi_3^{(1)}(\mu_1(\tau_1),0,0)+o(\frac{1}{t_1t_2})\}d\tau_2\Big],$$

where we have replaced the integrand by the expression (8.32), and the integration contour $|\tau_2|=1$ by $|\tau_2|=R_2$ with R_2 sufficiently large (note that the integrand is regular for $|\tau_2|>1$); this is permitted by Cauchy's theorem and because the integrand is uniformly bounded in τ_1 with $|\tau_2|=1$.

Next note that for $|\tau_2|=1$,

$$\lim_{|\tau_2|\to\infty}\tau_1\tau_2\mu_3(\frac{1}{\tau_1\tau_2}) = \lim_{|\tau_2|\to\infty}\tau_1\tau_2E\{(\tau_1\tau_2)^{-\mathbf{m}_3}\}=Pr\{\mathbf{m}_3=1\}. \tag{8.36}$$

Hence it follows by integrating the last integral in (8.35) for $R_2\to\infty$ that : for $|\tau_1|<1$, $|\tau_2|<1$,

$$\left[\frac{1}{2\pi i}\right]^2 \int_{|\tau_1|=1}\int_{|\tau_2|=1} \frac{\tau_1-1}{\tau_1-t_1}\,\frac{\tau_2-1}{\tau_2-t_2}\left[\frac{\Phi(\mu_1(\tau_1),0,\mu_3(\frac{1}{\tau_1\tau_2}))}{\{\mu_1(\tau_1)-1\}\{\mu_3(\frac{1}{\tau_1\tau_2})-1\}}+\frac{\Phi(\mu_1(\tau_1),0,0)}{\mu_1(\tau_1)-1}\right]d\tau_1,d\tau_2 = \tag{8.37}$$
$$-Pr\{\mathbf{m}_3=1\}\frac{1}{2\pi i}\int_{|\tau_1|=1}\frac{d\tau_1}{\tau_1}\,\frac{\tau_1-1}{\tau_1-t_1}\,\frac{\Phi_1(\mu_1(\tau_1),0,0)+\Phi_3^{(1)}(\mu_1(\tau_1),0,0)}{\mu_1(\tau_1)-1} =$$
$$Pr\{\mathbf{m}_3=1\}\Big[\frac{1}{t_1}\{\Phi(0,0,0)+\Phi_3^{(1)}(0,0,0)\}-\frac{t_1-1}{t_1}\{\frac{\Phi(\mu_1(t_1),0,0)+\Phi_3^{(1)}(\mu_1(t_1),0,0)}{\mu_1(t_1)-1}\}\Big].$$

Similarly, we obtain: for $|t_1|<1$, $|t_2|<1$,

$$\left[\frac{1}{2\pi i}\right]^2 \int_{|\tau_1|=1}\int_{|\tau_2|=1} \frac{\tau_1-1}{\tau_1-t_1}\,\frac{\tau_2-1}{\tau_2-t_2}\left[\frac{\Phi(0,\mu_2(\tau_2),\mu_3(\frac{1}{\tau_1\tau_2}))}{\{\mu_2(\tau_2)-1\}\{\mu_3(\frac{1}{\tau_1\tau_2})-1\}}+\frac{\Phi(0,\mu_2(t_2),0)}{\mu_2(\tau_2)-1}\right]d\tau_2 = \tag{8.38}$$
$$Pr\{\mathbf{m}_3=1\}\Big[\frac{1}{t_2}\{\Phi(0,0,0)+\Phi_3^{(1)}(0,0,0)\}-\frac{t_2-1}{t_2}\{\frac{\Phi(0,\mu_2(t_2),0)-\Phi_3^{(1)}(0,\mu_2(t_2),0)}{\mu_2(t_2)-1}\}\Big],$$

$$\left[\frac{1}{2\pi i}\right]^2 \int_{|\tau_1|=1}\int_{|\tau_2|=1} \frac{\tau_1-1}{\tau_1-t_1}\,\frac{\tau_2-1}{\tau_2-t_2}\left[\frac{\Phi(0,0,\mu_3(\frac{1}{\tau_1\tau_2}))}{\mu_3(\frac{1}{\tau_1\tau_2})-1}+\Phi(0,0,0)\right]=$$

$$Pr\{\mathbf{m}_3=1\}\{\Phi(0,0,0)+\Phi_3^{(1)}(0,0,0)\}.$$

Consequently, from (8.29), (8.34), (8.37) and (8.38), for $|t_1|<1$, $|t_2|<1$,

$$(t_1-1)(t_2-1)\frac{\Phi(\mu_1(t_1),\mu_2(t_2),0)}{\{\mu_1(t_1)-1\}\{\mu_2(t_2)-1\}} = Pr\{\mathbf{m}_3=1\}[(1+\frac{1}{t_1}+\frac{1}{t_2})\{\Phi(0,0,0)+\Phi_3^{(1)}(0,0,0)\} \tag{8.39}$$
$$-\frac{t_1-1}{t_1}\frac{\Phi(\mu_1(t_1),0,0)+\Phi_3^{(1)}(\mu_1(t_1),0,0)}{\mu_1(t_1)-1}-\frac{t_2-1}{t_2}\frac{\Phi(0,\mu_2(t_2),0)+\Phi_3^{(1)}(0,\mu_2(t_2),0)}{\mu_2(t_2)-1}],$$

and by continuity it is seen that the relations (8.49) also holds for $t_1=1$ and $t_2=1$.

By taking $t_1=1$ in (8.39) it follows from (8.26): for $|t_2|\leqslant 1$,

$$\frac{\Phi(1,0,0)}{E\{\mathbf{m}_1\}} = [(2+\frac{1}{t_2})\{\Phi(0,0,0)+\Phi_3^{(1)}(0,0,0)\}-\frac{1}{E\{\mathbf{m}_1\}}\{\Phi(1,0,0)+\Phi_3^{(1)}(1,0,0)\} \tag{8.40}$$
$$-\frac{t_2-1}{t_2}\frac{\Phi(0,\mu_2(t_2),0)+\Phi_3^{(1)}(0,\mu_2(t_2),0)}{\mu_2(t_2)-1}]Pr\{\mathbf{m}_3=1\},$$

and the analogous expression is obtained by taking $t_2=1$ in (8.39). Inserting these expressions in (8.49) shows that the left-hand side of (8.39) is independent of t_1 and also of t_2 for $|t_1|\leqslant 1$, $|t_2|\leqslant 1$. Hence for $|t_1|\leqslant 1$, $|t_2|\leqslant 1$,

$$(t_1-1)(t_2-1)\frac{\Phi(\mu_1(t_1),\mu_2(t_2),0)}{\{\mu_1(t_1)-1\}\{\mu_2(t_2)-1\}} = \Phi(0,0,0), \tag{8.41}$$

where the constant in the right-hand side of (8.51) follows by taking $t_1=t_2=0$.

From (8.41) it follows by taking $t_2=0$; for $|t_1|\leqslant 1$,

$$\Phi(\mu_1(t_1),0,0) = \frac{\mu_1(t_1)-1}{t_1-1}\Phi(0,0,0). \tag{8.42}$$

By similar arguments as those which have led to (8.27) we obtain: for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$\Phi(p_1,0,0) = \frac{(1-p_1)\phi_1(p_1)}{\phi_1(p_1-p_1)}\Phi(0,0,0), \tag{8.43}$$
$$\Phi(p_1,p_2,0) = \frac{(1-p_1)\phi_1(p_1)}{\phi_1(p_1-p_1)}\frac{(1-p_2)\phi_2(p_2)}{\phi_2(p_2-p_2)}\Phi(0,0,0),$$

and by symmetry we obtain the expressions for $\Phi(0,p_2,0)$, $\Phi(p_1,0,p_2)$, etc. Inserting these expressions in the right-hand side of (8.10)iii, and using the norming condition (8.10)i, proves Theorem 8.2 for the case $N=3$.

Using induction and the same type of arguments as used in the proof for $N=3$ leads to the proof of Theorem 8.2 for general N. □

9. The reflecting random walk with associated jump vector

In this section we consider the random walk $\mathbf{y}_n$, $n=0,1,2,\ldots$, on S with nonabsorbing boundary B, i.e. for $n=0,1,\ldots$,

$$\mathbf{y}_{n+1} = [\mathbf{y}_n-1]^+ + \boldsymbol{\eta}_n, \tag{9.1}$$
$$\mathbf{y}_0 = y_0,$$

where $\boldsymbol{\eta}_n$, $n=0,1,\ldots$, is a sequence of i.i.d. stochastic vectors; with $\boldsymbol{\eta}$ a generic element of this sequence we denote the N-variate generating function by

$$\omega(p) = E\{p^{\boldsymbol{\eta}}\}, \quad |p|=1, \quad \boldsymbol{\eta}\in S. \tag{9.2}$$

If $\boldsymbol{\eta}$ is associated with a stochastic vector $\boldsymbol{\xi}$, cf. Definition 6.1, then we call $\mathbf{y}_n$, $n=0,1,\ldots$, an *associated* random walk. Only the case that ξ has independent components and for which (3.11) holds will be considered here, so that,

$$E\{p^{\xi}\} = \phi(p)=\prod_{j=1}^{N}\phi_j(p_j), \quad (p_j)\leqslant 1, \quad j=1,\dots,N. \tag{9.3}$$

This implies, cf. def. (6.1) and (5,2), . . . , (5,6), that the vector $\hat{p}$ with

$$\hat{p}_j = \mu_j(t_j), \quad |t_j|=1, \quad \prod_{j=1}^{N} t_j=1, \quad j=1,\dots,N, \tag{9.4}$$

is a zero tuple of

$$p^1-E\{p^{\eta}\}, \quad |p|\leqslant 1. \tag{9.5}$$

It will further be assumed that

$$E\{\eta_j\}<1, \quad j=1,\dots,N. \tag{9.6}$$

REMARK 9.1. It suffices to assume that $E\{\eta_1\}<1$, because Theorem 7.1 then implies (9.6) for all $j=1,\dots,N$.

THEOREM a 9.1. *The random walk* $\mathbf{y}_n$, $n=0,1,\dots$, *defined by (9.1), . . . , (9.5) and with* $E\{\eta_j\}<1$, $j=1,\dots,N$, *has a unique stationary distribution; and with* $\mathbf{y}$ *a stochastic variable having this distribution, the N-variante generating function*

$$\Omega(p) = E\{p^{\mathbf{y}}\}, \quad |p|\leqslant 1, \tag{9.7}$$

is given by

$$\frac{\Omega(p)}{\Omega(0)} = \frac{\phi(p)-p^1}{\omega(p)-p^1}\,\frac{\omega(p)}{\phi(p)}\,\frac{\Phi(p)}{\Phi(0)}, \quad |p|\leqslant 1, \tag{9.8}$$

$$\Omega(0) = \frac{1-E\{\eta_1\}}{1-E\{\xi_1\}}\Phi(0), \tag{9.9}$$

with

$$\Phi(0) = \prod_{i=1}^{N} E\{1-\xi_i\}, \tag{9.10}$$

$$\Phi(p) = \prod_{j=1}^{N}\Big[\frac{(1-p_j)\phi_j(p_j)}{\phi_j(p_j)-p_j}E\{1-\xi_j\}\Big], \quad |p|\leqslant 1. \tag{9.11}$$

PROOF. Note that $\Phi(p)$, $|p|\leqslant 1$ is the N-variate generating of the stationary distribution of the random walk $\mathbf{x}_n$ discussed in Section 8, cf. Theorem 8.2.

As in Section 8, see the derivation of (8.9), it is shown by starting from (9.1) that $\Omega(p)$ should satisfy

$$[p^1-E\{p^{\eta}\}]\Omega(p) = (p-1)^1 E\{p^{\eta}\}\sum_{b\in\mathfrak{N}}\frac{\Omega(pb)}{(pb-1)^1}, \quad |p|\leqslant 1, \tag{9.12}$$

and for any zero tuple $\hat{p}$ of $p^1-E\{p^{\eta}\}$, $|p|\leqslant 1$, it should hold that

$$(\hat{p}-1)^1\sum_{b\in\mathfrak{N}}\frac{\Omega(\hat{p}b)}{(\hat{p}b-1)^1} = 0. \tag{9.13}$$

Since $\hat{p}$ as given by (9.4) is such a zero tuple and $\Omega(p)$ is by definition regular in any $|p_j|<1$, continuous in $|p_j|\leq 1$, the other p_i kept fixed, $|p_i|\leq 1$, it is readily seen that (9.13) and these regularity condidtions are identical with the conditions mentioned in (8.11)ii. In particular it is seen, cf. also the proof of Theorem 8.2 for $N=3$, that these conditions determine $\Omega(pb)$, $b\in\mathfrak{N}$ uniquely, and, actually, we have

$$\Omega(p_1,\ldots,p_{N-1},0) = \{\prod_{j=1}^{N-1}\frac{(1-p_j)\phi_j(p_j)}{\phi_j(p_j)-p_j}\}\Omega(0,0,0), \quad |p_j|\leq 1, \quad j=1,\ldots,N-1. \tag{9.14}$$

From (9.11), (9.12) and (9.14) the relation (9.8) follows directly. The norming condition i.e. $\Omega(1)=1$ yields, since $E\{\eta_j\}<1$, directly the relation (9.9). It should be noted that the construction of the relations (9.8) is unique so, since $0<\Omega(0,0,0)<1$ it follows that the y_n-process is positive recurrent with stationary distribution given by (9.8). □

COROLLARY 9.1. *For the condition of Theorem 9.1: every zero tuple $\hat{p}$ of $p^1-E\{p^{\eta}\}$, $|p|\leq 1$ is a zero tuple of $p^1-E\{p^{\xi}\}$, $|p|\leq 1$, except possibly a $\hat{p}$ with $\hat{p}^1=0$.*

PROOF. Because $\Omega(p)$, $|p|\leq 1$ is regular in each of its variables p_j with $|p_j|\leq 1$, and continuous in $|p_j|\leq 1$, the other variables kept fixed, it is seen from (9.10) that a zero tuple $\hat{p}$ of $p^1-E\{p^{\eta}\}$, $|p|\leq 1$, should be also a zero tuple of

$$[\phi(p)-p^1]\omega(p)\{\prod_{j=1}^{N}\frac{1-p_j}{\phi_j(p_j)-p_j}\}. \tag{9.15}$$

The last factor in (9.15) has no zero tuples in $|p|\leq 1$, so a zero tuple $\hat{p}$ of $\omega(p)-p^1$, $|p|\leq 1$ is a zero tuple of $\phi(p)-p^1$, $|p|\leq 1$ and/or of $\omega(p)$. But if $\omega(\hat{p})=0$ then necessarily $\hat{p}^1=0$, (it is however possible that also $\phi(\hat{p})=0$). The proof is complete. □

With ξ and η as defined above, cf. (9.1), . . . , (9.6), and

$$\xi_n=\{\xi_n^{(1)},\ldots,\xi_n^{(N)}\}, \quad \eta_n = \{\eta_n^{(1)},\ldots,\eta_n^{(N)}\}, \tag{9.16}$$

define for $n=1,2,\ldots$; $k=1,\ldots,N$,

$$\mathrm{s}_n^{(k)} := \sum_{m=1}^{n}(\xi_m^{(k)}-1), \quad \mathrm{t}_n^{(k)} := \sum_{m=1}^{n}(\eta_m^{(k)}-1). \tag{9.17}$$

The condition (3.11)ii implies that: for $|p_j|=1, p_j\neq 1, j=1,\ldots,N$,

$$|\frac{\phi(p)}{p^1}|<1, \quad |\frac{\omega(p)}{p^1}|<1, \tag{9.18}$$

so that for $|p_j|=1, p_j\neq 1, j=1,\ldots,N$,

$$\begin{aligned}\log\{1-\frac{\phi(p)}{p^1}\} &= -\sum_{n=1}^{\infty}\frac{1}{n}\{\frac{\phi(p)}{p^1}\}^n = -\sum_{n=1}^{\infty}\frac{1}{n}E\{p^{\mathrm{s}_n}\}= \\ &-\sum_{n=1}^{\infty}\frac{1}{n}E\{\prod_{k=1}^{N}p_k^{\mathrm{s}_n^{(k)}}[(\mathrm{s}_n^{(k)}\geq 0)+(\mathrm{s}_n^{(k)}<0)]\}.\end{aligned} \tag{9.19}$$

Analogously for

$$\log\{1-\frac{\omega(p)}{p^1}\},$$

and, consequently, for $|p_j|=1, p_j\neq 1, j=1,\ldots,N$,

$$\log\frac{1-\phi(p)/p^1}{1-\omega(p)/p^1} = -\sum_{n=1}^{\infty}\frac{1}{n}E\{\prod_{k=1}^{N}p_k^{\mathrm{s}_n^{(k)}}[(\mathrm{s}_n^{(k)}\geq 0)+(\mathrm{s}_n^{(k)}<0)]$$

$$-\prod_{k=1}^{N} p_k^{t_n^{(k)}}[(t_n^{(k)}\geqslant 0)+(t_n^{(k)}<0)]\}. \tag{9.20}$$

Since the left-hand side in (9.19) is also finite for $p_j=1$, $j=1,\ldots,N$, it follows by continuity that (9.19) also holds for $p_1=1$, $j=1,\ldots,N$. Note that it is known from Fluctuation Theory, cf. [6], that

$$E\{\xi_j<1\}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n}E\{s_n^{(j)}\geqslant 0\}<\infty, \tag{9.21}$$

$$E\{\eta_j<1\}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n}E\{t_n^{(j)}\geqslant 0\}<\infty,$$

so that the right-hand side of (9.20) is finite for $|p|=1$. It then follows from Theorem 7.3 and (9.20) that the left-hand side of (9.19) is regular in each p_j with $|p_j|<1$ and continuous in $|p_j|\leqslant 1$.

Hence by applying the operator

$$\{\prod_{j=1}^{N}\frac{1}{2\pi i}\int_{|p_j|=1}\frac{dp_j}{p_j-q_j}\}\cdots \quad \text{with} \quad |q_j|<1,\quad j=1,\ldots,N,$$

to the relation (9.19), it follows easily by contour integration that for $|p_j|\leqslant 1$,

$$\log\frac{1-\phi(p)/p^1}{1-\omega(p)/p^1} = -\sum_{n=1}^{\infty}\frac{1}{n}E\{\prod_{k=1}^{N}p_k^{s_n^{(k)}}(s_n^{(k)}\geqslant 0)-\prod_{k=1}^{N}p_k^{t_n^{(k)}}(t_n^{(k)}\geqslant 0)\}. \tag{9.22}$$

By letting $p\rightarrow 1$ it follows from (9.21) that

$$\frac{1}{1-E\{\xi_j\}}e^{-\sum_{n=1}^{\infty}\frac{1}{n}E\{\prod_{k=1}^{N}(s_n^{(k)}\geqslant 0)\}} = \frac{1}{1-E\{\eta_j\}}e^{-\sum_{n=1}^{\infty}\frac{1}{n}E\{\prod_{k=1}^{N}(t_n^{(k)}\geqslant 0)\}}. \tag{9.23}$$

REMARK 9.2. The relations (9.22) and (9.23) are of interest because they display the relation between the generating function of ξ and that of a with ξ associated variable for the case that ξ has independent components. It is, however, conjectured that this latter condition is not essential for the validity of the relations (9.22) and (9.23), and similarly for the validity of Theorem 9.1. Note that this theorem is based on Theorem (8.2)ii.

10. On the construction of the distribution of an associated random variable

In this section we shall describe in some detail the construction of the distribution of a variable η associated with ξ, with ξ satisfying the condition (3.11). The construction is based on the ideas descibed in Section 6. To expose the construction we shall discuss several examples.

First we consider the case that $N=3$ and

$$E\{p^{\xi}\} = \prod_{j=1}^{3}\phi_j(p_j),\quad |p_j|\leqslant 1,\quad j=1,2,3. \tag{10.1}$$

We start from the relation (6.4), i.e. for $p_k\in S_k^+\cup S_k$, $k=1,2$,

$$\sum_{b\in\mathfrak{N}_{b_3=0}} a_b E\{p^{\mathbf{k}_b}|\mathbf{n}_b<n\} = E\{[\frac{p_1}{\phi_1(p)}]^{\mathbf{m}_1-\mathbf{m}_3}[\frac{p_2}{\phi_2(p_2)}]^{\mathbf{m}_2-\mathbf{m}_3}(\mathbf{m}_1\geqslant\mathbf{m}_3)(\mathbf{m}_2\geqslant\mathbf{m}_3)\}. \tag{10.2}$$

By taking $p_1=0$ and/or $p_2=0$ we obtain for $p_k\in S_k^+\cup S_k$, $k=1,2$,

$$a_{000} = E\{(\mathbf{m}_1=\mathbf{m}_3)(\mathbf{m}_2=\mathbf{m}_3)\}, \tag{10.3}$$

$$a_{100}E\{p_1^{\mathbf{k}_{100}^{(1)}}|\mathbf{n}_{100}<\infty\} = -a_{000}+E\{[\frac{p_1}{\phi_1(p_1)}]^{\mathbf{m}_1-\mathbf{m}_3}(\mathbf{m}_1\geqslant\mathbf{m}_3)(\mathbf{m}_2=\mathbf{m}_3)\}$$

$$=E\{[\frac{p_1}{\phi_1(p_1)}]^{\mathbf{m}_1-\mathbf{m}_3}(\mathbf{m}_1>\mathbf{m}_3)(\mathbf{m}_2>\mathbf{m}_3)\},$$

$$a_{010}E\{p_2^{\mathbf{k}_{010}^{(1)}}|\mathbf{n}_{010}<\infty\} = E\{[\frac{p_2}{\phi_2(p_2)}]^{\mathbf{m}_2-\mathbf{m}_3}(\mathbf{m}_1=\mathbf{m}_3)(\mathbf{m}_2>\mathbf{m}_3)\},$$

$$a_{110}E\{p_1^{\mathbf{k}_{110}^{(1)}}p_2^{\mathbf{k}_{110}^{(2)}}|\mathbf{n}_{110}<\infty\} = E\{[\frac{p_1}{\phi_1(p_1)}]^{\mathbf{m}_1-\mathbf{m}_3}E\{[\frac{p_2}{\phi_2(p_2)}]^{\mathbf{m}_2-\mathbf{m}_3}(\mathbf{m}_1>\mathbf{m}_3)(\mathbf{m}_2>\mathbf{m}_3)\},$$

and by symmetry

$$a_{001}E\{p_3^{\mathbf{k}_{001}^{(3)}}|\mathbf{n}_{001}<\infty\} = E\{[\frac{p_3}{\phi_3(p_3)}]^{\mathbf{m}_3-\mathbf{m}_1}(\mathbf{m}_3>\mathbf{m}_1)(\mathbf{m}_2=\mathbf{m}_1)\}, \tag{10.4}$$

$$a_{101}E\{p_1^{\mathbf{k}_{101}^{(1)}}p_3^{\mathbf{k}_{101}^{(3)}}|\mathbf{n}_{101}<\infty\} = E\{[\frac{p_1}{\phi_1(p_1)}]^{\mathbf{m}_1-\mathbf{m}_2}E\{[\frac{p_3}{\phi_3(p_3)}]^{\mathbf{m}_3-\mathbf{m}_2}(\mathbf{m}_1>\mathbf{m}_2)(\mathbf{m}_3>\mathbf{m}_2)\},$$

$$a_{011}E\{p_2^{\mathbf{k}_{011}^{(2)}}p_3^{\mathbf{k}_{011}^{(3)}}|\mathbf{n}_{011}<\infty\} = E\{[\frac{p_2}{\phi_2(p_2)}]^{\mathbf{m}_2-\mathbf{m}_1}[\frac{p_3}{\phi_3(p_3)}]^{\mathbf{m}_3-\mathbf{m}_1}(\mathbf{m}_2>\mathbf{m}_1)(\mathbf{m}_3>\mathbf{m}_1)\}.$$

We next consider the case with

$$\phi_3(p_3) = c_0+c_1p_3, \quad |p_3|\leqslant 1, \quad c_0+c_1=1, \quad 0<c_1<1. \tag{10.5}$$

Note that (10.5) violates (3.11)iii, however, since the discussion below actually concerns only hitting points the violation of (3.11)iii is here irrelevant.

As in Section 5 we write for a zero tuple of

$$p^1-E\{p^{\mathbf{\xi}}\}, \quad |p|\leqslant 1, \tag{10.6}$$

for $|t_j|=1, \quad j=1,2,3; \quad t_1t_2t_3 = 1,$ (10.7)

$$\hat{p}_1 = \mu_1(t_1) = E\{t_1^{\mathbf{m}_1}\}, \quad \hat{p}_2 = \mu_2(t_2)=E\{t_2^{\mathbf{m}_2}\},$$

$$\hat{p}_3 = \mu_3(t_3) = E\{t_3^{\mathbf{m}_3}\}=\frac{c_0t_3}{1-c_1t_3},$$

the expression for $\hat{p}_3$ follows directly from (5.2) and (10.5).

For the present case the hitting point identity reads, cf. also (6.3), for $|t_j|=1, j=1,2,$

$$\mu_1(t_1)\mu_2(t_2)\frac{c_0}{t_1t_2-c_1} = \sum_{b\in\mathfrak{N}} a_bE\{\hat{p}^{\mathbf{k}_b}|\mathbf{n}_b<\infty\}=E\{\hat{p}^{\mathbf{k}}\}. \tag{10.8}$$

From (10.8) we obtain for $|t_1<1, |t_2|<1,$

$$\sum_{b\in\mathfrak{N}_{b_3=0}} a_bE\{\hat{p}^{\mathbf{k}_b}|\mathbf{n}_b<\infty\} = \left[\frac{1}{2\pi i}\right]^2\int_{|\tau_1|=1}\int_{|\tau_2|=1}\frac{d\tau_1}{\tau_1-t_1}\;\frac{d\tau_2}{\tau_2-t_2}\mu_1(\tau_1)\mu_2(\tau_2)\frac{c_0}{\tau_1\tau_2-c_1}= \tag{10.9}$$

$$= c_0\sum_{n=0}^{\infty}c_1^n\frac{1}{2\pi i}\int_{|\tau_1|=1}\frac{d\tau_1}{\tau_1-t_1}E\{\tau_1^{\mathbf{m}_1-n-1}\}\frac{1}{2\pi i}\int_{|\tau_2|=1}\frac{d\tau_2}{\tau_2-t_2}E\{\tau_2^{\mathbf{m}_2-n-1}\}$$

$$= c_0\sum_{n=0}^{\infty}c_1^nE\{t_1^{\mathbf{m}_1-n-1}(\mathbf{m}_1\geqslant n+1)\}E\{t_2^{\mathbf{m}_2-n-1}(\mathbf{m}_2\geqslant n+1)\}=$$

$$= c_0\sum_{n=0}^{\infty}c_1^n\frac{E\{t_1^{\mathbf{m}_1}\}-E\{t_1^{\mathbf{m}_1}(\mathbf{m}_1\leqslant n)\}}{t_1^{n+1}}\;\frac{E\{t_2^{\mathbf{m}_2}\}-E\{t_2^{\mathbf{m}_2}(\mathbf{m}_2\leqslant n)\}}{t_2^{n+1}},$$

and it is readily seen that (10.9) also holds for $|t_1|=1, |t_2|=1.$

Next we evaluate the expressions (10.4) for $\phi_3(p_3)$ as given by (10.5). We start from: for $|t_1|<1,$ $|t_3|<1,$

$$E\{t_1^{\mathbf{m}_1-\mathbf{m}_2}t_3^{\mathbf{m}_3-\mathbf{m}_2}(\mathbf{m}_1\geqslant\mathbf{m}_2)(\mathbf{m}_3\geqslant\mathbf{m}_2)\}= \tag{10.10}$$

$$\left[\frac{1}{2\pi i}\right]^2 \int_{|\tau_1|=1} \int_{|\tau_3|=1} \frac{d\tau_1}{\tau_1 - t_1} \frac{d\tau_3}{\tau_3 - t_3} \mu_1(\tau_1)\mu_2(\frac{1}{\tau_1\tau_3})\mu_3(\tau_3) =$$

$$\left[\frac{1}{2\pi i}\right]^2 \int_{|\tau_1|=1} \int_{|\tau_3|=1} \frac{d\tau_1}{\tau_1 - t_1} \frac{d\tau_3}{\tau_3 - t_3} \frac{c_0\tau_3}{1-c_1\tau_3} \mu_1(\tau_2)\mu_2(\frac{1}{\tau_1\tau_3}) =$$

$$\frac{c_0 c_1}{1-c_1 t_3} \frac{1}{2\pi i} \int_{|\tau_1|=1} \frac{d\tau_1}{\tau_1 - t_1} E\{\tau_1^{\mathbf{m}_1-\mathbf{m}_2} c_1^{\mathbf{m}_2}\} =$$

$$\frac{c_0 c_1}{1-c_1 t_3} E\{t_1^{\mathbf{m}_1-\mathbf{m}_2} c_1^{\mathbf{m}_2} (\mathbf{m}_1 \geqslant \mathbf{m}_2)\}.$$

Hence for $|t_3|<1$,

$$E\{t_3^{\mathbf{m}_3-\mathbf{m}_2} (\mathbf{m}_1=\mathbf{m}_2)(\mathbf{m}_3 \geqslant \mathbf{m}_2)\} = \frac{c_0 c_1}{1-c_1 t_3} E\{c_1^{\mathbf{m}_2} (\mathbf{m}_1=\mathbf{m}_2)\}. \tag{10.11}$$

Hence from (10.10) and (10.11) for $|t_1|<1$, $|t_3|<1$,

$$E\{t_1^{\mathbf{m}_1-\mathbf{m}_2} t_3^{\mathbf{m}_3-\mathbf{m}_2} (\mathbf{m}_1>\mathbf{m}_2)(\mathbf{m}_3 \geqslant \mathbf{m}_2)\} = \frac{c_0 c_1}{1-c_1 t_3} E\{t_1^{\mathbf{m}_1-\mathbf{m}_2} c_1^{\mathbf{m}_2} (\mathbf{m}_1>\mathbf{m}_2)\}. \tag{10.12}$$

And from (10.12) we obtain for $|t_1|<1$, $|t_3|<1$,

$$E\{t_1^{\mathbf{m}_1-\mathbf{m}_2} t_3^{\mathbf{m}_3-\mathbf{m}_2} (\mathbf{m}_1>\mathbf{m}_2)(\mathbf{m}_3>\mathbf{m}_2)\} = \frac{c_0 c_1^2 t_3}{1-c_1 t_3} E\{t_1^{\mathbf{m}_1-\mathbf{m}_2} c_1^{\mathbf{m}_2} (\mathbf{m}_1>\mathbf{m}_2)\}. \tag{10.13}$$

Consequently we obtain from (10.4),(10.10), . . . , (10.13) by using also a symmetry argument, and by noting that (10.11), . . . , (10.13) also holds for $|t_1|=1$, $|t_3|=1$,

$$a_{101} E\{\hat{p}_1^{\mathbf{k}_{101}^{(1)}} \hat{p}_3^{\mathbf{k}_{101}^{(3)}} | \mathbf{n}_{101}<n\} = \frac{c_0 c_1^2 t_3}{1-c_1 t_3} E\{t_1^{\mathbf{m}_1-\mathbf{m}_2} c_1^{\mathbf{m}_2} (\mathbf{m}_1>\mathbf{m}_2)\}, \tag{10.14}$$

$$a_{011} E\{\hat{p}_2^{\mathbf{k}_{011}^{(2)}} \hat{p}_3^{\mathbf{k}_{011}^{(3)}} | \mathbf{n}_{011}<n\} = \frac{c_0 c_1^2 t_3}{1-c_1 t_3} E\{t_2^{\mathbf{m}_2-\mathbf{m}_1} c_1^{\mathbf{m}_1} (\mathbf{m}_2>\mathbf{m}_1)\},$$

$$a_{001} E\{\hat{p}_3^{\mathbf{k}_{001}^{(1)}} \hat{p}_3^{\mathbf{k}_{001}^{(3)}} | \mathbf{n}_{001}<n\} = \frac{c_0 c_1^2 t_3}{1-c_1 t_3} E\{c_1^{\mathbf{m}_2} (\mathbf{m}_1=\mathbf{m}_2)\}.$$

Next we apply the inverse mappings as defined in (6.3), and obtain from (10.9) and (10.14) using the notation

$$M_{jk} := Pr\{\mathbf{m}_j=k\}, \quad j=1,2; \quad k=1,2,\ldots,$$
$$:= 0, \qquad j=1,2; \quad k=0,$$

for $p_j \in S_j^+ \cup S_j$, $j=1,2,3,\ldots$, cf. (10.8),

$$E\{p^{\mathbf{k}}\} = \sum_{b\in\mathcal{K}} a_b E\{\hat{p}^{\mathbf{k}_b} | \mathbf{n}_b<\infty\} \tag{10.15}$$

$$= c_0 \sum_{n=0}^{\infty} c_1^n \left[\frac{\phi(p_1)\phi_2(p_2)}{p_1 p_2}\right]^{n+1} \{p_1 - \sum_{k=0}^{n} M_{1k} [\frac{p_1}{\phi_1(p_1)}]^k\}\{p_2 - \sum_{k=0}^{n} M_{2k} [\frac{p_2}{\phi_2(p_2)}]^k\}$$

$$c_1^2 p_3 [E\{[\frac{p_1}{\phi_1(p_1)}]^{\mathbf{m}_1-\mathbf{m}_2} c_1^{\mathbf{m}_2} (\mathbf{m}_1>\mathbf{m}_2)\} + E\{[\frac{p_2}{\phi_2(p_2)}]^{\mathbf{m}_2-\mathbf{m}_1} c_1^{\mathbf{m}_1} (\mathbf{m}_2>\mathbf{m}_1)\} + E\{c_1^{\mathbf{m}_2} (\mathbf{m}_1=\mathbf{m}_2)\}].$$

Conseqently by taking $|p_3|=1$ we have for $p_j \in S_j^+ \cup S_j$, $j=1,2$,

$$\omega(p_1^{\mathbf{k}_1} p_2^{\mathbf{k}_2}) = c_0 \sum_{n=0}^{\infty} c_1^n \left[\frac{\phi(p_1)\phi_2(p_2)}{p_1 p_2}\right]^{n+1} \{p_1 - \sum_{k=0}^{n} M_{1k} [\frac{p_1}{\phi_1(p_1)}]^k\}\{p_2 - \sum_{k=0}^{n} M_{2k} [\frac{p_2}{\phi_2(p_2)}]^k\} \tag{10.16}$$

$$+c_1^2[E\{[\frac{p_1}{\phi_1(p_1)}]^{\mathbf{m}_1-\mathbf{m}_2}c_1^{\mathbf{m}_2}(\mathbf{m}_1>\mathbf{m}_2)+[\frac{p_2}{\phi_2(p_2)}]^{\mathbf{m}_2-\mathbf{m}_1}c_1^{\mathbf{m}_1}(\mathbf{m}_2>\mathbf{m}_1)+c_1^{\mathbf{m}_2}(\mathbf{m}_1=\mathbf{m}_2)\}],$$

or with replacing $\mathbf{k}$ by η, it follows that $\eta_1=(\eta_1,\eta_2)$ with generating function: for $p_j \in S_j^+ \cup S_j$, $j=1,2$,

$$\omega(p_1,p_2) = E\{p_1^{\eta_1}p_2^{\eta_2}\} =$$

$$\phi_1(p_1)\phi_2(p_2)c_0\sum_{n=0}^{\infty}c_1^n\frac{\phi_1^n(p_1)-\sum_{k=1}^{n}M_{1k}p_1^k\phi_1^{n-k}(p_1)}{p_1^n}\,\frac{\phi_2^n(p_2)-\sum_{k=2}^{n}M_{2k}p_2^k\phi_2^{n-k}(p_2)}{p_2^n}+$$

$$c_1^2E\{[\frac{p_1}{\phi_1(p_1)}]^{\mathbf{m}_1-\mathbf{m}_2}c_1^{\mathbf{m}_2}(\mathbf{m}_1>\mathbf{m}_2)+[\frac{p_2}{\phi_2(p_2)}]^{\mathbf{m}_2-\mathbf{m}_1}c_1^{\mathbf{m}_1}(\mathbf{m}_2>\mathbf{m}_1)+c_2^{\mathbf{m}_2}(\mathbf{m}_1>\mathbf{m}_2)]\}, \tag{10.17}$$

is associated with $\xi=\{\xi_1,\xi_2\}$, with

$$E\{p^{\xi}\} = \phi_1(p_2)\phi_2(p_2), \quad |p_1|\leqslant 1,\ |p_2|\leqslant 1.$$

The expression (10.17) shows clearly the composition of the generating function of the variable $\eta=(\eta_1,\eta_2)$ associated with $\xi=(\xi_1,\xi_2)$, viz. the terms of the sum stem from the situation that the first entrance into B is at the set $B_{110}\cup B_{100}\cup B_{010}\cup B_{000}$, the other terms in (10.17) stem from a first entrance into $B_{001}\cup B_{011}\cup B_{101}$.

A further insight in (10.17) is obtained by taking c_1 small, i.e. $0<c_1<<1$, so that : for $c_1\downarrow 0$,

$$\omega(p_1,p_2) = (1-c_1)\phi_1(p_1)\phi_2(p_2)\{1+c_1\frac{\phi_1(p_1)-\phi_1(0)p_1}{p_1}\,\frac{\phi_2(p_2)-\phi_2(0)p_2}{p_2}+o(c_1)\} \tag{10.18}$$

in particular

$$E\{\eta_1\} = (1-c_1)[-c_1+\{1-c_1\phi_1(0)+2c_1\}E\{\xi\}]+o(c_1), \tag{10.19}$$

$$= (1-c_1)[-c_1\{1-E\{\xi_1\}\}+E\{\xi_1\}(1-c_1\phi_1(0)+c_1)]+o(c_1)<E\{\xi_1\}+o(c_1).$$

The construction of a distribution of a variable η_1 associated with ξ discussed above is based on the idea described in Section 6, iii with $N=3$; obviously, a similar, construction can be given for general N.

Next we consider a construction based on convexity. We take $N=2$ and

$$E\{p^{\xi}\} = E\{p_1^{\xi_1}p_2^{\xi_2}\}=\phi_1(p_1)\phi_2(p_2), \quad |p_1|\leqslant 1, \quad |p_2|\leqslant 1. \tag{10.20}$$

Let α_1, α_2, α_3 be constants with: for $i=1,2,3,\ldots$,

$$0\leqslant\alpha_i\leqslant 1, \quad \alpha_1+\alpha_2+\alpha_3=1, \tag{10.21}$$

and let for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, $\mathfrak{N}=(00,01,10)$,

$$\psi(p_1,p_2) := \sum_{b\in\mathfrak{N}}a_bE\{p^{\mathbf{k}_b}|\mathbf{n}_b<\infty\}= \tag{10.22}$$

$$= a_{00}+a_{10}E\{p_1^{\mathbf{k}_{10}^{(1)}}|\mathbf{n}_{10}<\infty\}+a_{01}E\{p_2^{\mathbf{k}_{01}^{(2)}}|\mathbf{n}_{01}<\infty\},$$

be the generating function of the distribution of the hitting point of the boundary B of the absorbing random walk of Section 2 with $N=2$ and with the distribution of ξ given by (10.20), cf. the discussion below (6.6).

Define the 2-variate generating function $\omega(p_1,p_2)$ of the vector $\eta=(\eta_1,\eta_2)$ by: for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$\omega(p_1,p_2) = E\{p_1^{\eta_1}p_2^{\eta_2}\} = \alpha_1p_1p_2+\alpha_2\phi_1(p_1)\phi_2(p_2)+\alpha_3\phi(p_1,p_2). \tag{10.23}$$

Let $\hat{p}_1$, $\hat{p}_2$ be a zero tuple of the kernel

$$p_1p_2-\phi_1(p_1)\phi_2(p_2),\quad |p_1|\leqslant 1,\quad |p_2|\leqslant 1, \tag{10.24}$$

it then follows immediately from (10.32), from Theorem 4.2 and from (10.21) that $\hat{p}_1, \hat{p}_2$ is also a zero tuple of

$$p_1p_2-\omega(p_1,p_2),\quad |p_1|\leqslant 1,\quad |p_2|\leqslant 1; \tag{10.25}$$

and hence the vector $\eta=(\eta_1,\eta_2)$ with 2-variate generating function given by (10.23) is associated with ξ.

In the next section we shall discuss an interesting application of the associated vector η with generating function given by (10.23).

Finally we present another example of an associated random variable. Again we take

$$E\{p_1^{\xi_1}p_2^{\xi_2}\} = \phi_1(p_1)\phi_2(p_2),\quad |p_1|\leqslant 1,\quad |p_2|\leqslant 1. \tag{10.26}$$

With

$$0\leqslant\alpha_0\leqslant 1,\quad 0\leqslant\alpha_1\leqslant 1,\quad \alpha_1+\alpha_2=1, \tag{10.27}$$

we put for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$\omega(p_1,p_2) := \phi_1(p_1)\phi_2(p_2)[\alpha_0+\alpha_1\frac{\phi_1(p_1)-\phi_1(0)}{p_1}\frac{\phi_2(p_2)-\phi_2(0)}{p_2}]+ \tag{10.28}$$

$$\alpha_1\phi_1(0)\phi_2(p_2)+\alpha_1\phi_2(0)\phi_1(p_2)-\alpha_1\phi_1(0)\phi_2(0).$$

It is readily seen that a zero tuple $\hat{p}_1, \hat{p}_2$ of

$$p_1p_2-\phi_1(p_1)\phi_2(p_2),\quad |p_1|\leqslant 1,\quad |p_2|\leqslant 1,$$

is also a zero tuple of

$$p_1p_2-\omega(p_1,p_2),\quad |p_1|\leqslant 1,\quad |p_2|\leqslant 1,$$

and that

$$\omega(1,1) = 1. \tag{10.29}$$

It remains to show $\omega(p_1,p_2)$ is a bivariate generating function of a stochastic vector $\eta=(\eta_1,\eta_2)$ with state space $\{0,1,2,\dots\}\times\{0,1,2,\dots\}$, i.e., the coefficients of $p_1^hp_2^k$, $h,k\in\{0,1,2,\dots\}$ in the series expansion of $\omega(p_1,p_2)$ should all be nonnegative. Actually, this is readily verified, from (10.28), note that $\omega(0,0)=\phi(0,0)$.

Simple algebra shows that (10.28) may be rewritten as: for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, $0<\alpha_1\leqslant 1$,

$$\omega(p_1,p_2) = p_1p_2+[\phi_1(p_1)\phi_2(p_2)-p_1p_2][1+\alpha_1\frac{\phi_1(p_1)-\phi_1(0)}{p_1}\frac{\phi_2(p_2)-\phi_2(0)}{p_2}]. \tag{10.30}$$

11. On a simple associated random walk

In this section we shall consider in some detail the random walk $\{y_n, n=0,1,2,\dots\}$ as defined in Section 9 for the case $N=2$ and with the distribution of the vector $\eta=(\eta_1,\eta_2)$ given by: for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$.

$$\omega(p_1,p_2) = E\{p_1^{\eta_1}p_2^{\eta_2}\}=\alpha_1p_1p_2+\alpha_2\phi_1(p_1)\phi_2(p_2), \tag{11.1}$$

$$\phi_i(p_i) = E\{p_i^{\xi_i}\},\quad i=1,2;\quad \xi_i\in\{0,1,2,\dots\}.$$

$$\phi(p_1,p_2) = \phi_1(p_1)\phi_2(p_2),$$

$$0\leqslant\alpha_1<1,\quad 0<\alpha_2\leqslant 1,\quad \alpha_1+\alpha_2=1.$$

It follows readily that : for $|\hat{p}_1|\leq 1$, $|\hat{p}_2|\leq 1$,

$$\phi_1(\hat{p}_1)\phi_2(\hat{p}_2)-\hat{p}_1\hat{p}_2=0 \Leftrightarrow \omega(\hat{p}_1,\hat{p}_2)-\hat{p}_1\hat{p}_2=0, \tag{11.2}$$

so that η is associated with $\xi=(\xi_1,\xi_2)$ and conversely.

We shall first consider some relations between the moments of η and ξ.

Put

$$c_i := E\{\eta_i\}, \quad i=1,2, \tag{11.3}$$
$$d := E\{(\eta_1-c_1)(\eta_2-c_2)\}.$$

It is readily seen, cf. also Theorem (7.1), that

$$1-E\{\eta_i\} = \alpha_2\{1-E\{\xi_i\}\}, \quad i=1,2, \tag{11.4}$$
$$1-E\{\eta_i^2\} = \alpha_2\{1-E\{\xi_i^2\}\}, \quad i=1,2,$$
$$d = E\{(\eta_1-c_1)(\eta_2-c_2)\}=\alpha_1\alpha_2E\{1-\xi_1\}E\{1-\xi_2\}.$$

From now on we shall only consider the case that

$$0<c_i<1, \quad 0<E\{\xi_i\}<1, \quad i=1,2. \tag{11.5}$$

This leads to some restriction on α_2 besides those in (11.1).

From (11.4) and (11.5) it follows that

$$\alpha_2 > \max(1-c_1,\ 1-c_2), \tag{11.6}$$
$$\alpha_2 > \max(E\{\eta_1^2\},\ E\{\eta_2^2\}),$$
$$\alpha_2 = \frac{(1-c_1)(1-c_2)}{(1-c_1)(1-c_2)+d}.$$

Note that (11.4) and (11.5) imply that

$$0<d<\tfrac{1}{4}, \tag{11.7}$$

and that (11.6) implies

$$0<d<-c_1c_2+\min(c_1,c_2). \tag{11.8}$$

Denote by $\{\mathrm{x}_n,\ n=0,1,2,\dots\}$ the random walk as defined in Section 9 with $N=2$ and the generating function of ξ as given in (11.1).

The condition (11.5) implies that both the y_n and the x_n random walks possess unique stationary distributions. Let y and x be stochastic vectors having these distributions, respectively, and put: for $|p_1|\leq 1$, $|p_2|\leq 1$,

$$\Omega(p_1,p_2) = E\{p_1^{\mathrm{y}_1}p_2^{\mathrm{y}_2}\}, \quad \Phi(p_1,p_2) = E\{p_1^{\mathrm{x}_1}p_2^{\mathrm{x}_2}\}. \tag{11.9}$$

By using Theorem 9.1 is follows from (9.11) and (11.1) that: for $|p_1|\leq 1$, $|p_2|\leq 1$,

$$\frac{\Omega(p_1,p_2)}{\Omega(0,0)} = \frac{1}{\alpha_2}\{\alpha_1p_1p_2+\alpha_2\phi_1(p_1)\phi_2(p_2)\}\frac{1-p_1}{\phi_1(p_1)-p_1}\,\frac{1-p_2}{\phi_2(p_2)-p_2}, \tag{11.10}$$

or by using

$$\Omega(1,1) = 1,$$

$$\Omega(p_1,p_2) = \{\alpha_1p_1p_2+\alpha_2\phi_1(p_1)\phi_2(p_2)\}\frac{1-p_1}{\phi_1(p_1)-p_1}\,\frac{1-p_2}{\phi_2(p_2)-p_2}\,E\{1-\xi_1\}E\{1-\xi_2\}. \tag{11.11}$$

From (11.11) it follows that for $i=1,2$,

$$E\{y_i\} = \alpha_1+\alpha_2 E\{\xi_i\}+\frac{\frac{1}{2}E\{\xi_i(\xi_i-1)\}}{1-E\{\xi_i\}} = E\{x_i\}+\alpha_1 E\{1-\xi_i\}, \tag{11.12}$$

$$\mathrm{cov}(y_1,y_2) = \alpha_1\alpha_2 E\{1-\xi_1\}E\{1-\xi_2\}=d.$$

Note that

$$\mathrm{cov}(x_1,x_2) = 0. \tag{11.13}$$

$$E\{x_i\}-E\{\xi_i\} = \frac{1}{2}\frac{E\{\xi_i(\xi_i-1)\}}{1-E\{\xi_i\}} = \frac{1}{2}\frac{E\{\eta_i(\eta_i-1)\}}{1-E\{\eta_i\}} = E\{y_i\}-E\{\eta_i\}.$$

The relations derived above, in particular (11.12) and (11.13), indicate clearly the relations between random walks generated by associated jump vectors: η and ξ. From a view point of practical application it is of interest to investigate whether it is possible to approximate the distribution of a jump vector η by a distribution of a jump vector ξ such that η is (approximately) associated with ξ; because if the stationary distribution of the random walk generated by ξ exists and can be determined then much information about the random walk generated by η can be obtained.

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